Dimensional Analysis

CUED 1A Lecture notes

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with examples from the lecture notes of
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Chapter 1

Introduction to dimensional analysis

When performing quantitative calculation in the physical sciences, students at the 10+2 level around the world (the A-levels here in the UK) are taught to track the units and ensure that they are consistent with the quantity they are calculating. Underlying the consistency of units is, in fact, a fundamental principle of physics. The objective of dimensional analysis is to reveal the underlying principle and its myriad applications.

1.1 Motivating examples

To motivate the discussion, consider the following way to approach some questions.

1. Let us start with a simple one – vibrations of a stretched string.

   Question 1.1. A guitar string has a mass of 10 g and length 0.25 m. When a tension of 100 N is applied to stretch it, what is the fundamental frequency of its natural vibrations?

   Answer 1.1. The units of the answer we seek, the frequency of vibrations (f), is Hz, i.e. 1/s. Noting that 1 N = 1 kg m/s^2, there is only one way to combine the units of tension, mass and length to get the units of 1/s, viz.

   $$f = C\sqrt{\frac{100 \text{ kg m/s}^2}{0.25 \text{ m} \times 0.01 \text{ kg}}} = 200C \ \frac{1}{\text{s}}, \tag{1.1}$$

   where C is a constant that does not have any units. This constant C cannot be predicted in this method, and so this approach appears to be futile. But a moment's thought reveals that the method produces another result. It produces the manner in which the independent parameters (the tension, the mass and length of the string) *must* appear in this calculation for frequency. So, if the tension were τ, the string mass and length were m and l, respectively, then the frequency of its natural vibration would be related to it through the following relation:

   $$f = C\sqrt{\frac{\tau}{ml}}. \tag{1.2}$$

   In this case the exact answer is

   $$f = \frac{1}{2}\sqrt{\frac{\tau}{ml}}, \tag{1.3}$$

   so C is the constant 1/2.

   (Did you notice that we presumed that the frequency depends on and only on the tension, the string mass and length?)

2. Consider astronomy for the next example.

   Question 1.2. An extrasolar planet is observed to orbit its host star in 10^6 s. The star has a mass of 10^{30} kg. Estimate the distance of the planet from the star. The gravitational force of attraction is quantified by the gravitational constant $G = 6.67 \times 10^{-11}$ m^3/(kg s^2). Assume a circular orbit.

Answer 1.2. We know that the relation between the period of orbit and the radius of orbit does not involve the mass of the planet itself, but depends on the mass of the host star. We wish to determine the radius, R, which has SI units of m. There is only one combination of the period, $T = 10^6$ s, the stellar mass, $M = 10^{30}$ kg, and G,

$$R = C\left(6.67 \times 10^{-11}\frac{\mathrm{m}^3}{\mathrm{kg\ s}^2} \times 10^{30}\ \mathrm{kg} \times (10^6\ \mathrm{s})^2\right)^{1/3} = 4.05C \times 10^{10}\ \mathrm{m}, \tag{1.4}$$

where C is a constant without units. Again, without knowing C, we cannot evaluate R, but we seem to have discovered how this radius depends on the stellar mass, the gravitational constant and the orbital period. Indeed,

$$R = C \times (GMT^2)^{1/3}. \tag{1.5}$$

Note that we arrived at this expression without any knowledge of orbital mechanics. Using orbital mechanics, one can determine $C = 1/(4\pi^2)^{1/3}$.

3. Let us now consider an example from astronomy for which we may not know how to compute the answer.

Question 1.3. A black hole has such a strong gravitational pull that even light cannot escape it's vicinity. The edge of this region is called the event horizon. Estimate the radius of the event horizon for a supermassive black hole of mass $M = 10^{39}$ kg.

Answer 1.3. The gravitational force is characterized by the mass of the black hole and the gravitational constant $G = 6.67 \times 10^{-11}$ m^3/(kg s^2). Light is characterized by its speed, $c = 3 \times 10^8$ m/s. The event horizon radius, R, has units of m, which can be constructed only by the combination

$$R = C\frac{6.67 \times 10^{-11}\ \mathrm{m}^3/(\mathrm{kg\ s}^2)\ \times 10^{39}\ \mathrm{kg}}{(3 \times 10^8\ \mathrm{m/s})^2} = 7.41 \times 10^{11}C\ \mathrm{m}. \tag{1.6}$$

Again, C is a constant, which is indeterminate without the application of the theory of general relativity. Yet, for any black hole, we can deduce that the event horizon radius is given by

$$R = C\frac{GM}{c^2}. \tag{1.7}$$

This equation implies that a black hole of twice the mass has twice as large an event horizon. And we deduced that without using general relativity.

4. Let us now look at an example, from the realm of quantum mechanics, where we may not know another way of obtaining the answer.

Question 1.4. The energy of an electron orbiting a proton in a hydrogen atom consists of its electrostatic part and its kinetic part. The electrostatic energy depends on the charge of the electron, $e = 1.6 \times 10^{-19}$ C, the permittivity of free space, $\epsilon_0 = 8.85 \times 10^{-12}$ C^2 s^2/ (kg m^3), the electronic mass $m = 9.1 \times 10^{-31}$ kg, and the distance between the distance between the electron and the proton. Quantum mechanics says that the energy is quantized, i.e. it can take on only discrete values. It further introduces uncertainty about where the electron's position. This uncertainty is quantified by the Planck constant, $h = 6.62 \times 10^{-34}$ kg m^2/s. Estimate the electrostatic energy of the electron in its ground state, i.e. lowest discrete energy level.

Answer 1.4. The units of energy are kg m^2/s^2. Let us presume that the electrostatic energy of the electron in its ground state depends on e, ϵ_0, h and m. It is a little tricky to see but there is only one way to combine these parameters to make the units of energy, but here it is.

$$E = C \times \frac{(9.1 \times 10^{-31}\ \mathrm{kg})\ (1.6 \times 10^{-19}\ \mathrm{C})^4}{(6.62 \times 10^{-34}\ \mathrm{kg\ m}^2/\mathrm{s})^2\ (8.85 \times 10^{-12}\ \mathrm{C}^2\ \mathrm{s}^2/(\mathrm{kg\ m}^3))^2} = 1.73 \times 10^{-17}C\frac{\mathrm{kg\ m}^2}{\mathrm{s}^2}. \tag{1.8}$$

Here, again, C is a constant without units, which cannot be determined without application of quantum mechanics. Regardless, the ground state electrostatic energy of the electron depends on the properties of the hydrogen atom as

$$E = C\frac{me^4}{h^2\epsilon_0^2}. \tag{1.9}$$

All we used in these examples is that the units match and without using any knowledge of the underlying physics. Indeed, general relativity and quantum mechanics are very advanced topics outside the scope of this module, but they all must respect units. The numerical constant C cannot be determined simply by using the units of quantities, but do not let that disappoint you. For example, a single experimental measurement of the frequency of a vibrating string under tension can yield the C for Question 1.1.

Here is another example.

Question 1.5. (from Dr John P Longley's Lecture Notes)

Two point particles of masses m_1 and m_2, respectively, travel along a frictionless straight horizontal track with speeds $v_1 > 0$ and v_2, respectively. They collide, adhere and travel with a new speed v_3. Graphing v_3 in terms of m_1, m_2, v_1 and v_2 is the objective of this exercise. Draw a single comprehensible graph that summarises all possible collisions.

Answer 1.5. Applying conservation of momentum $(m_1 + m_2)v_3 = m_1v_1 + m_2v_2$ gives us v_3 as

$$v_3 = \frac{m_1v_1 + m_2v_2}{m_1 + m_2}. \tag{1.10}$$

This expression has v_3 depend on four variables (m_1, v_1, m_2 and v_2), so it is not obvious how a single graph can summarise all possible cases. The space of variables is five dimensional, whereas the page has only two dimensions. If we plot v_3 against, say, v_1 for different values of v_2, m_1 and m_2, the graph will be cluttered beyond comprehension!

However, we can rearrange equation (1.10) as follows

$$\frac{v_3}{v_1} = \frac{1 + \left(\frac{m_2}{m_1}\right)\left(\frac{v_2}{v_1}\right)}{1 + \left(\frac{m_2}{m_1}\right)}. \tag{1.11}$$

Equation (1.11) while being completely equivalent to equation (1.10), only involves three quantities (m_1/m_2), (v_2/v_1) and (v_3/v_1), so all possible collisions could be plotted as a series of curves, one each for a given values of (m_2/m_1) on a set of axes of (v_3/v_1) against (v_2/v_1). Such a graph is shown below.

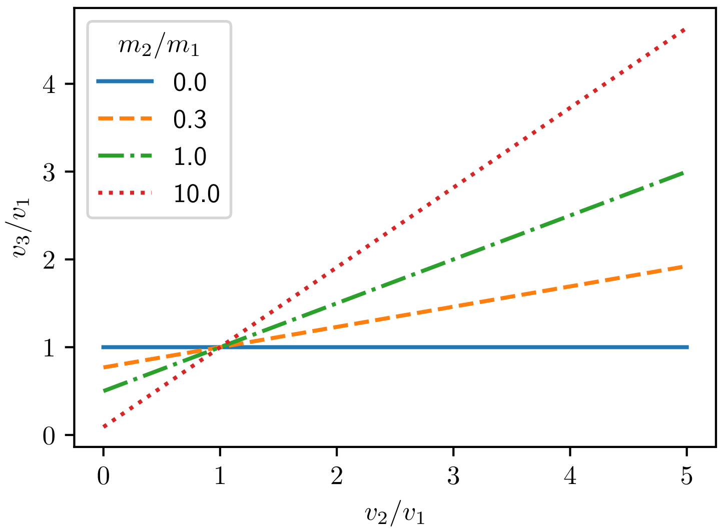

Figure 1.1: A single graph summarizing the result of equation (1.10).

Question 1.5 demonstrates that the number of parameters required to represent a phenomenon can be reduced by judicious re-organization of the parameters.

1.2 Objectives and outline

These examples showed a few application in an *ad hoc* manner of the principle that units must match, but the principle is so much more powerful. A systematic study of the consequences of this principle is called *dimensional* *analysis*. The primary application of dimensional analysis lies in reducing the number of parameters required to describe a phenomenon, as shown in Question 1.5. Fewer experiments need to be performed to span the parameter space that is reduced using dimensional analysis. When a scale model of a wing, a car, an aeroplane or a rocket is tested in a wind tunnel, it is dimensional analysis at work behind the scenes.

Here we develop this subject in the following order:

1. Units,
   (a) SI Units,
   (b) Scales and systems of units,
   (c) Conversion between units,
2. Dimensions,
   (a) Definition of base and derived dimensions,
   (b) Dimensional consistency,
   (c) Dimensionless parameters,
3. Dimensionless forms of equations
   (a) Physical relations between dimensionless parameters,
   (b) Buckingham's Pi theorem,
4. Applications,
   (a) Dimensionless representation of experimental data, and
   (b) Scale models.

1.3 Conclusion

The principle of matching of units is a profound one, which can be exploited to our advantage if treated systematically. The aims of the course are as follows (from Dr John P Longley's Lecture Notes)
:

- To introduce and illustrate the use of dimensional analysis.
- To develop an understanding of the principles of dimensional consistency.
- To develop techniques required to form non-dimensional quantities and relationships.
- To explain how dimensional analysis can be used to simplify problems by reducing the number of parameters.
- To correlate experimental data and to assist in the design of models for testing using dimensional principles.

Chapter 2

Units

Physical quantities possess units. For example, a meter is a unit of length, a kilogram is a unit of mass and a second is a unit of time. (There are also other units for length, mass and time, which we will discuss later in this chapter.) This principle transcends the various realms of physics – a meter is a unit of length in classical physics, general relativistic physics, and in quantum physics, even though the notions of length itself is not the same in these realms. The origin of units for physical quantities is, therefore, somewhat mysterious. (When you have difficulty falling asleep, ponder about the origin of units for physical quantities.)

Advanced tip. The function served by units for physical quantities, to an extent, is also served by units defined for certain mathematical (or even unphysical) quantities. For example, it is sometimes useful to describe the number of apples in a basket and/or the number of bananas with the units of apples and bananas. In economics and commerce, pounds (£) and dollars ($) are units of currency, which are not based in any physical laws, and these units also serve many of the purposes described in this modules. Once the underlying principles are understood, they can also be applied to abstract mathematical problems.

I think the reason quantities have units is that those quantities are not interchangeable or even comparable with others (without suitable conversion, which we will learn about in this chapter). A meter cannot be interchanged with a second or a kilogram. A dollar is not interchangeable with a pound or an apple without an appropriate conversion factor. This seems like a very fundamental property.

Some mathematicians have already attempted to quantify Romeo's love for Juliet (e.g. see here, here and, if all else fails, here). Love for Juliet is not interchangeable with meters, kilograms, seconds, apples, or dollars. So rest assured that even love (when expressed mathematically) has units. (Disclaimer: I neither approve nor disapprove of mathematically quantifying love. Please exercise own judgment in this matter.)

The focus in this module is, of course, on physical quantities, and the main function of units is to standardize measurement of the physical quantity. We start with the definitions of the *Le Système International d'Unités*, which is French for the International System of Units, and is abbreviated as SI units.

2.1 SI units for physical quantities

The SI system of units is the standard that most of the world has adopted for representing physical quantities. The system consists of the "**Base units**" shown in Table 2.1.

Quantity	Unit	Symbol
Mass	kilogram	kg
Length	metre	m
Time	second	s
Electric Current	Ampere	A
Absolute Temperature	Kelvin	K
Luminous Intensity	candela	cd
Amount of substance	kilomole	kmol

Table 2.1: The **base units** in the SI system of units.

From these base units, all other units are derived based on definition of the derived quantities or based on some physical law. Here are two examples.

Question 2.1. Determine the SI units of velocity based on the definition as length per unit time and acceleration defined as rate of change of velocity.

Answer 2.1. The derived units are constructed as below.

Quantity	**Unit**	**Symbol**
Velocity = Length/Time	meters/second	m/s
Acceleration = Velocity/Time	$\frac{\text{(meters/second)}}{\text{second}}$	$\frac{\text{m/s}}{\text{s}}$
	= meters/second^2	= m/s^2

Question 2.2. Determine the SI units of force, work and power. Here you may use Newton's second law of motion (force = mass × acceleration), and the definitions of work (force × distance) and power (work per unit time).

Answer 2.2. The derived unit is constructed as below.

Quantity	**Unit**	**Symbol**
Force	kilogram metres/second^2 = Newton	kg m/s^2 = N
Work	Newton metres = Joule	N m = J
Power	Joule/second = Watt	J/s = W

Advanced tip. The definition of units has evolved from being based on physical artefacts to being based on physical constants. For example, from approximately 1800 to 1927, the definition of a metre was based on two fine marks made on an platinum-iridium bar stored in Paris, France. This artefact-based definition has disadvantages. The artefacts are not easy to reproduce nor can they be easily adapted for every kind of measurement. These disadvantages caused an evolution of the artefacts themselves and ultimately, in 1927 for the metre, a shift to artefact-free definition. Today, the metre is defined in terms of the speed of light (which is a physical constant) and the definition of a second. Here is one resource that provides all the information on the SI units and their history.

2.2 Scales and systems of units

As we are all familiar, the metre is not the only unit of length, so is a yard, a foot, a furlong and a light year. Each unit of length is related to the other units of length by a scale factor. This also holds for all quantities. For example,

$$1 \text{ inch} = 0.0254 \text{ metres}, \qquad 1 \text{ mile} = 1760 \text{ yards} = 5280 \text{ feet}, \tag{2.1}$$

$$1 \text{ feet} = 12 \text{ inches}, \qquad 1 \text{ hour} = 3600 \text{ seconds}. \tag{2.2}$$

These conversion factors define an identity in the following way:

$$\frac{1 \text{ feet}}{12 \text{ inches}} = 1, \qquad \frac{1 \text{ hour}}{3600 \text{ seconds}} = 1. \tag{2.3}$$

Identities such as (2.3) allow us to convert from one units to another. (Note that there are no conversions between a metre and a second – a length cannot be expressed in units of time.)

Advanced tip. An exception to this rule of scale factor is temperature. The conversion of temperature from Celsius to Kelvin uses a shift

$$\text{temperature in K} = \text{temperature in }^\circ\text{C} + 273.15 \text{ K}. \tag{2.4}$$

Temperature is a quantity for which, in some instances, it is beneficial to set the zero by shifting the scale. This is in addition to the scale factor discussed above. That is how the Fahrenheit scale is related to the Celsius

$$\text{temperature in }^\circ\text{F} = \text{temperature in }^\circ\text{C} \times \frac{9}{5} + 32^\circ\text{F} \tag{2.5}$$

A shift is permissible for scales of temperature but not for other units because in many situations involving temperature, it is only temperature differences that matter (and it is only in those situations that a shifted scale can be used). Note that temperature differences convert from one unit to another purely using scale factors, e.g.

$$\text{Temperature difference in }{}^\circ\text{F} = \frac{9}{5} \times \text{Temperature difference in }{}^\circ\text{C}. \tag{2.6}$$

The scale of absolute temperature is necessary for situations such as the ideal gas law

$$PV = nRT \tag{2.7}$$

where P is the pressure of a gas, V is the volume it occupies, n is the amount of the gas (e.g. in moles), R is the gas constant, and T is the *absolute* temperature. Values of temperature expressed in a shifted scale such as Celsius or Fahrenheit cannot be used in the ideal gas law.

A few words now about a few other common systems of units apart from the SI system in historical and present prevalence.

2.2.1 CGS system

The base units in this system are the centimeter (length), gram (mass) and second (time).

2.2.2 MKS mass system

Also known simply as the MKS system, this is the system prevalent before the adoption of the SI system. The base units of length, mass and time are the metre, kilogram and the second.

2.2.3 MKS force system

This is the first example we see where the base quantities are force, length, and time. The base units of these quantities are kilogram force (kgf), metre and second, respectively. A kgf is defined as the weight of 1 kilogram mass in the MKS mass system.

2.2.4 British mass system a.k.a the Imperial system

The base units of mass, length and time in this system are the pound (mass), foot and second.

2.2.5 American Engineering (force) System

Analogous to the MKS force system, the base quantities in this system are force, length and time, with units pound (force), foot and second. A pound force (abbreviated as lbf for clarity) is defined as the weight of the pound mass (abbreviated as lbm for clarity).

2.3 Conversion of derived units

The identity involving scale factors can be used to convert derived units from one system of units to another. The following examples illustrate how.

Question 2.3. (from Dr John P Longley's Lecture Notes)
A typical car is 14 ft long. How long is it in metres?

Answer 2.3.

$$14\text{ ft} = 14\text{ ft} \times \frac{0.3048\text{ m}}{1\text{ ft}} = 4.267\text{ m}. \tag{2.8}$$

Question 2.4. (from Dr John P Longley's Lecture Notes)
A typical car speed is 35 mph. How fast is that in m/s?

Answer 2.4.

$$35\text{ mph} = 35\text{ mph} \times \frac{5280\text{ ft}}{1\text{ mile}} \times \frac{0.3048\text{ m}}{1\text{ ft}} = 15.65\text{ m/s}. \tag{2.9}$$

Note the use of multiple conversion factors to arrive at the desired units.

2.4 Conclusion

In this chapter, we introduced units, scales of units and conversion between units.

Chapter 3

Dimensions

Units are defined by humans whereas dimensions arise from nature. The property of a quantity to be represented by a set of inter-convertible units is termed as its *dimension*. The radius of the earth is 6400 km = 3976 miles = 31814 furlongs = 7×10^6 yards, while the mass of earth is 5.9×10^{24} kg = 1.3×10^{25} lbm. The independence of the units of these two quantities is encapsulated by assigning them a dimension. The earth's radius has dimensions of length and the mass has dimensions of mass. All units of length form a so-called equivalence class (that is the technical term) by virtue of being convertible between themselves and so do the units of mass. The concept of dimensions thus removes the human-dependent definition of the system of units and abstracts the equivalence-independence relationship between units of physical quantities.

3.1 Notation

Dimensions are symbolically denoted by a letter, as shown in table 3.1.

Quantity	**SI unit**	**Dimensions**
Mass	kg	M
Length	m	L
Time	s	T
Electric current	A	I
Temperature	K	Θ

Table 3.1: Base dimensions in the (nominal) MLT system, along with their SI units.

Advanced tip. The concept and notation for dimensions was first explicated by James Clerk Maxwell in 1871, who was then the Cavendish Professor of Physics at Cambridge. Maxwell enclosed the symbol for dimension within square brackets, so that $[L]$ denoted the dimension of length instead of simply L (this convention is followed by many to date). The reason was to not conflate the dimension $[L]$ with any symbol L that perhaps denoted some length. This notation with the square brackets is still used today in many sources to denote dimensions.

3.2 Base and derived dimensions

The property of base and derived units carries over to dimensions. The symbols in table 3.1 denote the MLT system, where mass, length and time form the base dimensions. Dimensions of derived quantities are determined in the same way as for derived units.

Question 3.1. Determine the dimensions of velocity, acceleration, force in the MLT system of dimensions.

Answer 3.1. L/T, L/T^2 and ML/T^2, respectively (see table 3.2 for more).

3.3 System of dimensions

Systems of dimensions are distinguished by the choice of base dimensions. Just as the MLT system, one may define the FLT system to mirror the MKS force system of units, where the base dimensions are force (F), length (L) and time (T). In such a system, the derived quantities are derived in table 3.3.

Quantity		Dimensions
Velocity	length/time	L/T
Acceleration	rate of change of velocity	L/T^2
Force	mass times acceleration	ML/T^2
Density	mass per unit volume	M/L^3
Momentum	mass times velocity	ML/T
Work	Force times distance	ML^2/T^2
Power	Rate of work	ML^2/T^3
Pressure	Force per unit area	$M/(LT^2)$

Table 3.2: Some derived dimensions in the MLT system.

Quantity		Dimensions
Velocity	length/time	L/T
Acceleration	rate of change of velocity	L/T^2
Mass	force divided by acceleration	FT^2/L
Density	mass per unit volume	FT^2/L^4
Momentum	mass times velocity	FT
Work	Force times distance	FL
Power	Rate of work	FL/T
Pressure	Force per unit area	F/L^2

Table 3.3: Some derived dimensions in the FLT system.

3.4 Specification of dimensional quantities

(from Dr John P Longley's Lecture Notes)
To completely specify any quantity that has dimensions it is necessary to give two pieces of information, the numerical value and the units in which it is measured, e.g.,

$$\text{time:} \qquad t = 2.5\ \text{s}, \qquad \text{or} \qquad \text{distance:} \qquad d = 35.4\ \text{km}. \tag{3.1}$$

A single space is always left between the value and the unit, and neither should be italicized.

3.5 Principle of dimensional consistency

This principle is the foundation upon which the structure of dimensional analysis is erected.

Principle 3.1. There are two parts:

1. The dimension of product (or ratio) of two quantities is the product (or ratio) of the dimensions.
2. Only quantities with identical dimensions may be added, subtracted, equated or compared with each other. This principle is also called the principle of dimensional homogeneity.

Note that we have been following principle 3.1-1 in constructing the derived dimensions. And we tacitly used principle 3.1-2 in Questions 1.1-1.4

Consider the following two examples as demonstration.

Question 3.2. (from Dr John P Longley's Lecture Notes)
Show that the equation for the period, t, of a simple pendulum as it depends on the length of the pendulum, l, and the acceleration due to gravity, g,

$$t = 2\pi\sqrt{\frac{l}{g}} \tag{3.2}$$

is dimensionally consistent.

Answer 3.2. The dimensions of t are T, of l are L and of g are L/T^2. Therefore, the dimensions of the two sides of the equation are

$$\text{LHS} = T, \qquad \text{RHS} = \sqrt{\frac{L}{L/T^2}} = T. \tag{3.3}$$

In determining the dimensions of the RHS, we employed principle 3.1-1 of dimensional consistency, and the dimensions of two sides being equal demonstrates principle 3.1-2.

Question 3.3. Show that the expression (1.3) for the natural frequency of oscillation of a stretched string is dimensionally consistent. For convenience, here is the equation again

$$f = \frac{1}{2}\sqrt{\frac{\tau}{ml}},$$

where f is the frequency, τ is the tension in the string, m is its mass and l its length.

Answer 3.3. The dimension of the LHS (frequency) are $1/T$. The dimensions of the tension τ (which is a force) are ML/T^2, while those of m are M and l are L. The dimensions of the RHS are to be determined according to principle 3.1-1 as

$$RHS = \sqrt{\frac{ML/T^2}{ML}} = \frac{1}{T} \quad .$$

Since both the dimensions of the LHS and RHS are $1/T$, expression (1.3) obeys principle 3.1-2 of dimensional consistency.

It is also useful to examine an example that violates dimensional consistency.

Question 3.4. In cardiology, the peak pressure difference, Δp, between the left ventricle and the aorta, measured in mm of Hg, is determined from the peak speed, v, of blood at the inlet of the aorta, measured in m/s, using the expression

$$\Delta p = 4v^2.$$

Is this expression dimensionally consistent?

Answer 3.4. The dimensions of the LHS are $M/(LT^2)$, whereas those of the RHS (determined using principle 3.1-1 of dimensional consistency) are L^2/T^2. The two sides have different dimensions and, therefore, violate principle 3.1-2 of dimensional consistency.

Now we come to the crux of the matter. Dimensional consistency implies that the equation looks the same in any consistent set of units.

> **Principle 3.2. Dimensional consistency:** A complete statement of a physical law is independent of the system of measurement.

In other words, the same expression relating multiple physical quantities holds in any consistent set of units. We state that the Principle 3.2 implies and is implied by Principle 3.1. (Interested reader may find the proof in "Dimensional Analysis and Theory of Models" by Henry L Langhar, John Wiley & Sons, Inc., New York, 1951.) Let us demonstrate this using equation (3.2) from Question 3.2, the example of the simple pendulum.

Question 3.5. (from Dr John P Longley's Lecture Notes)
Assume that the equation (3.2) holds in the SI system of units. Define a custom set of units using 1 arm-length = 0.8 m and 1 heart-beat = (2/3) s. Show that (3.2) also holds in these new sets of units.

Answer 3.5. Let us abbreviate arm-length to al and heart-beat to hb. So 1 al = 0.8 m and 1 hb = (2/3) s. The numerical value of t, g and l in the two sets of units will be different (owing to the conversion factors), and so we will denote those differently. Let t_{value}, l_{value} and g_{value} be the numerical values of t, l and g, respectively, in SI units. And let t'_{value}, l'_{value} and g'_{value} be them in the new units. Applying the conversion factors,

$$\begin{aligned}
t'_{\text{value}} &= t_{\text{value}} \times \frac{1\text{ hb}}{(2/3)\text{ s}} && = \frac{3}{2}\, t_{\text{value}}, \\
l'_{\text{value}} &= l_{\text{value}} \times \frac{1\text{ al}}{0.8s} && = \frac{5}{4}\, l_{\text{value}}, \\
g'_{\text{value}} &= g_{\text{value}} \times \frac{(2/3)^2\text{ s}^2}{1\text{ hb}^2} \times \frac{1\text{ al}}{0.8\text{ m}} && = \frac{5}{9}\, g_{\text{value}},
\end{aligned}$$

Now notice how the expression $\sqrt{l/g}$ transforms

$$\sqrt{\frac{l'_{\text{value}}}{g'_{\text{value}}}} = \sqrt{\frac{(5/4)\ l_{\text{value}}}{(5/9)\ g_{\text{value}}}} = \frac{3}{2}\sqrt{\frac{l_{\text{value}}}{g_{\text{value}}}}$$

exactly as t does (by a factor of (3/2)). Therefore, both

$$t_{\text{value}} = 2\pi\sqrt{\frac{l_{\text{value}}}{g_{\text{value}}}} \qquad \text{and} \qquad t'_{\text{value}} = 2\pi\sqrt{\frac{l'_{\text{value}}}{g'_{\text{value}}}} \tag{3.4}$$

hold. In general, for any simple pendulum that obeys equation (3.2), the values of the parameters can be substituted in any consistent system of units.

Question 3.6. Let us revisit the expression in question 3.3 for the expression for the natural frequency of a stretched string. Show that converting from SI to custom units for length (length-units, abbreviated lu), mass (mass units, abbreviated mu) and time (time-units, abbreviated tu) with (symbolic) conversion factors λ, μ and τ defined as

$$1\ \text{m} = \lambda\ \text{lu}, \qquad 1\ \text{kg} = \mu\ \text{mu}, \qquad \text{and} \qquad 1\ \text{s} = \tau\ \text{tu},$$

leaves the expression invariant.

Answer 3.6. Left as an exercise for the reader.

Why does dimensional consistency imply that an equation is independent of system of units? Without loss of generality, let us consider an equation described by the MLT set of base dimensions. The principle 3.1-2 of dimensional consistency implies that each term on the left and right hand side of the equation (or inequality) has the same dimensions, say $M^aL^bT^c$ for some exponents a, b and c. If μ, λ and τ are the conversion factors between two sets of units for mass, length and time respectively (just as in question 3.6), then each term in the equation will accumulate the same conversion factor of $\mu^a\lambda^b\tau^c$. (This factor follows from principle 3.1-1 of dimensional consistency. By definition, the conversion factors are non-zero.) Therefore, canceling this factor from each term recovers the same equation in the new system of units.

3.6 Dimensionless combinations of parameters

A combination of parameters that end up with dimensions of unity is called a dimensionless parameter, dimensionless group or dimensionless number. An excellent example of a dimensionless number is the Mach number. For an aircraft flying with speed v in a medium with the speed of sound c, its Mach number is the ratio v/c. Each of v and c are speeds, so have dimensions L/T. But the Mach number is a ratio of the two and therefore has dimensions of unity (in other words $M^0L^0T^0$).

Here are a few other examples.

Question 3.7. Show that the coefficient of static friction is dimensionless.

Answer 3.7. The coefficient of static friction (μ) is used to determine the maximum force of friction between static surfaces in contact using the equation

$$f = \mu n, \tag{3.5}$$

where f is the magnitude of the strongest force of friction and n is the normal force of compression between the two surfaces. Naturally, μ has the dimensions of the ratio of two forces, and is, therefore, dimensionless.

Ratio of quantities with identical dimensions is not the only way to get dimensionless numbers. Consider the following question.

Question 3.8. The coefficient of lift, C_L, of a wing generating a lift force (e.g. an airplane wing or a helicopter blade) is defined as the ratio

$$C_L = \frac{l}{\frac{1}{2}\rho v^2 a} \tag{3.6}$$

where ℓ is the lift force generated by the wing, ρ is the density of the medium, v is the speed of the wing relative to the medium and a is the area of the wing. Show that C_L is dimensionless. An F22 raptor weighs 30000 kg and has a wing area of 78 m^2. When cruising at a speed of 1482 km/hr at sea level, what is the lift coefficient?

Answer 3.8. The lift force in the numerator has dimensions of ML/T^2. The denominator derives it dimensions from $\rho v^2 a$, which are

$$\frac{M}{L^3} \times \left(\frac{L}{T}\right)^2 \times L^2 = \frac{ML}{T^2}.$$

The numerator and denominator both have dimensions of force, so their ratio is dimensionless.

For the F22 raptor (at sea level where the air density is 1.2 kg/m^3), the lift coefficient is

$$C_L = \frac{30000 \text{ kg} \times 9.8 \text{ m/s}^2}{\frac{1}{2} \times 1.2 \text{ kg/m}^3 \times \left(1482\frac{\text{km}}{\text{hr}} \times \frac{1000 \text{ m}}{1 \text{ km}} \times \frac{1 \text{ hr}}{3600 \text{ s}}\right)^2 \times 78 \text{ m}^2} = 0.0371. \tag{3.7}$$

Although the lift coefficient is not technically ratio of two forces, it may be written as the ratio of two quantities that have dimensions of force. A moment's thought will convince you that a dimensionless number can be expressed as a ratio of two forces, two lengths, two velocities, two times, or any two quantities whose dimensions can be constructed from the constituent dimensions of the parameters making up the dimensionless number. A conveniently constructed such ratio is usually used as an interpretation of the dimensionless number.

Question 3.9. (from Dr John P Longley's Lecture Notes)
Calculate the number of car-lengths travelled in a half-hour long car journey at 35 miles/hr. Assume the car is 14 ft long.

Answer 3.9.

$$\text{Ratio} = \frac{35\frac{\text{miles}}{\text{hr}} \times 0.5 \text{ hr}}{14 \text{ ft} \times \frac{1 \text{ mile}}{5280 \text{ ft}}} = 6.60 \times 10^3.$$

This number will be the same in any system of units because it is dimensionless.

Other examples of dimensionless numbers are constants (π, 17.3, e, etc), arguments and values of mathematical functions (e.g. both θ and $\sin\theta = \theta - \theta^2/3! + \theta^5/5! \ldots$).

Principle 3.3. A property of dimensionless numbers is that they have the same value in any (consistent) set of units, i.e. the conversion factor for these parameters between systems of units is unity (in the notation of Question 3.6, the factor is $\mu^0\lambda^0\tau^0 = 1$).

3.7 Conclusion

Dimensions abstract that property of quantities which enable them to be expressed in a set of inter-convertible units. When a physical law is independent of the system of units when expressed in a dimensionally consistent form. This implies that any complete statement of a physical law can be stated in terms of dimensionless variable alone. We will see in Chapter 5 the central role Dimensionless numbers play in dimensional analysis.

Can now do Examples Paper Q1-4.

Chapter 4

Dimensionless forms of equations

Many problems in engineering and physics, which involve equations relating one dependent quantity on several parameters, can be simplified using dimensional analysis, whether or not the precise form of the relationship is known. In most case, the simplification is achieved by expressing the problem in dimensionless terms. This core technique forms the basis of the applications of dimensional analysis in engineering and is explained in this chapter.

4.1 Physical relations between dimensionless parameters

A primitive element of dimensional analysis is the dependence of a quantity on one or more parameters of the problem. The relationship can be specified, such as in the example of an uniformly accelerating car. Consider a car moving at a velocity u, which is then accelerated at a constant rate a. We require the velocity v of the car after a time t, which is given by

$$v = u + at. \tag{4.1}$$

We know this equation from first principles and we can use such examples to glean the principles, which we can apply in situation where the form of the equation is unknown. When it is unknown, we will write such a dependence symbolically as

Parameter: v	depends on	$\{ u,$	$a,$	t	$\}$
Dimensions: LT^{-1}		$\{ LT^{-1},$	$LT^{-2},$	T	$\}$

which stands for $v = f(u, a, t)$ for some unspecified function $f()$. The second line of the equation lists the dimensions of the quantities above them.

We now state the consequence of dimensional consistency on physical laws.

Principle 4.1. In a complete statement of a physical law, it is possible to rearrange the terms so that all groups or quantities are dimensionless.

Question 1.5 is an example of this consequence. This result follows directly from the principles 3.2 in §3.5 and principle 3.3 in §3.6. We will further explore this consequence in this section.

4.2 Non-dimensionalizing equations

Here are a few more example.

Question 4.1. (from Dr John P Longley's Lecture Notes)
Consider a car moving at a (non-zero) velocity u which then accelerates at a constant rate a. The velocity after time t is $v = u + at$. Write this in a dimensionless form.

Answer 4.1. Let us begin by identifying the dimensions of the terms in the equations.

$$\begin{array}{lccccc} \text{Equation :} & v & = & u & + & at \\ \text{Dimensions :} & \dfrac{L}{T} & & \dfrac{L}{T} & & \dfrac{L}{T^2} \times T \end{array}$$

Let us divide this equation by u to get

$$\frac{v}{u} = 1 + \frac{at}{u}.$$

Both v/u and at/u are dimensionless (as is the number 1). This equation is equivalent to the original equation and consists only of dimensionless quantities.

Question 4.2. The range of a projectile shot with speed u at an angle θ on flat ground is

$$d = \frac{u^2 \sin 2\theta}{g}.$$

Express this equation using dimensionless numbers.

Answer 4.2. Dividing by d yields

$$\frac{u^2}{gd} \sin 2\theta = 1.$$

Here $u^2/(gd)$ is dimensionless, so is $\sin 2\theta$. Note that an equally acceptable expression is

$$\frac{gd}{u^2} = \sin 2\theta.$$

Any function of dimensionless numbers is also dimensionless.

Question 4.3. The trajectory of a projectile is given by the relations

$$x = ut, \qquad y = vt - \frac{1}{2}gt^2, \tag{4.2}$$

where u and v are the initial components of velocity in the x and y directions, respectively, and t is the time elapsed since launch. This equation holds until the projectile hits the ground, say, at $y = 0$. Express it in dimensionless terms.

Answer 4.3. The first instinct is to divide the x equation by ut and the y equation by vt, and doing so does yield a dimensionless form of the equation

$$\frac{x}{ut} = 1, \qquad \frac{y}{vt} = 1 - \frac{gt}{2v}. \tag{4.3}$$

Here, the dimensionless parameters are $x/(ut)$, $y/(vt)$ and $gt/(2v)$. However, sometimes, we wish to retain certain properties of the relation, for example in this case the linear and quadratic dependence on time. So consider dividing the x equation by uv/g (inspired by Question 4.2) and the y equation by v^2/g (because that is related to the maximum height reached by the projectile). Both uv/g and v^2/g have dimensions of L. This yields

$$\frac{gx}{uv} = \frac{gt}{v}, \qquad \frac{gy}{v^2} = \frac{gt}{v} - \frac{1}{2}\left(\frac{gt}{v}\right)^2. \tag{4.4}$$

This equation can be expressed in terms of dimensionless variables $\tilde{x} = gx/(uv)$, $\tilde{y} = gy/(v^2)$ and $\tilde{t} = gt/v$ in the form

$$\tilde{x} = t, \qquad \tilde{y} = \tilde{t} - \frac{1}{2}\tilde{t}^2. \tag{4.5}$$

We will discuss an advantage of (4.5) over (4.3) in the §4.3. Here $\tilde{x}$, $\tilde{y}$ and $\tilde{t}$ are considered to be the dimensionless versions of x, y and t, even though they also contain g, u and v.

Question 4.3 shows that constructing dimensionless forms of equations may not be a matter of simply dividing by one of the terms. And that there is no unique way to non-dimensionalize an equation. While many ways of non-dimensionalizing are technically correct and algebraically equivalent, some are more useful than others, because they reflect the user's interpretation of the equations. We will see more examples of this in §4.3.

4.3 Non-dimensionalizing charts

(The accelerating car example in this section and the solution upto and including Principle 4.2 is from Dr John P Longley's Lecture Notes.)

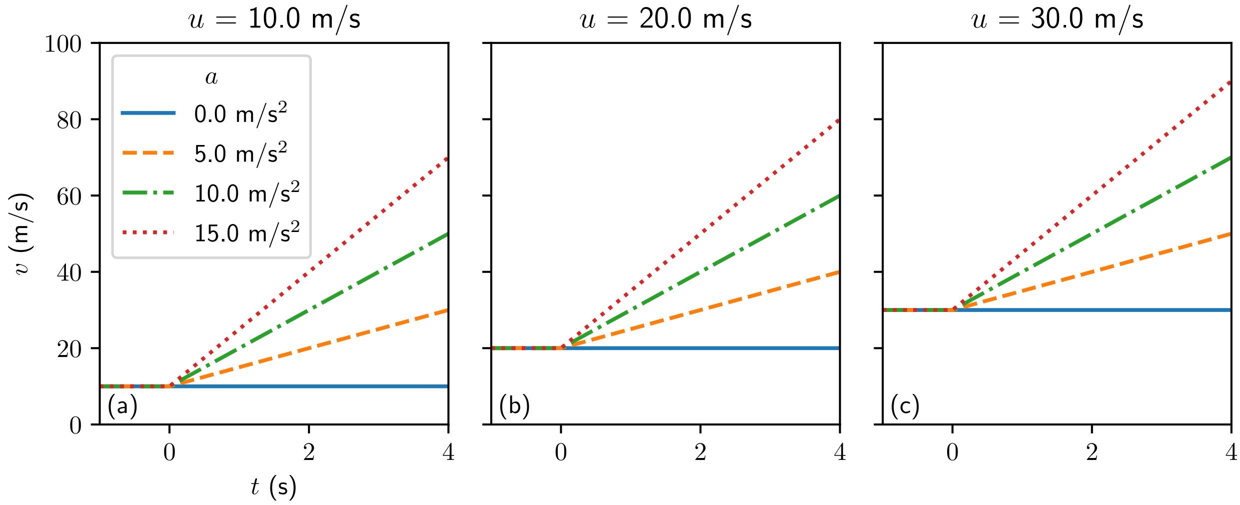

Figure 4.1: The velocity of a car as a function of time starting with speed u and accelerating at a rate a. The three panels show three initial speeds (a) $u = 10$ m/s, (b) $u = 20$ m/s, and (c) $u = 30$ m/s. Legend in panel (a) shows the acceleration a for each curve (the same legend applies to panels b and c).

It is possible that a relationship between variables cannot be expressed in terms of nice equations but is available in the form of charts. For our accelerating car example, we can plot v as a function of time t for various u and a. The velocity of a car as it accelerates starting from three different initial velocities and four different accelerations is shown in Figure 4.1.

For example the graphs in Figure 4.1(a) represents town driving with an initial velocity of 10 m/s and perhaps 4.1(c) represents motorway driving with $u = 30$ m/s. To present every possible initial velocity u (i.e. driving condition) would require an entire book of graphs!

As a first step, consider plotting the ratio v/u as a function of time, as is presented in Figure 4.2. When

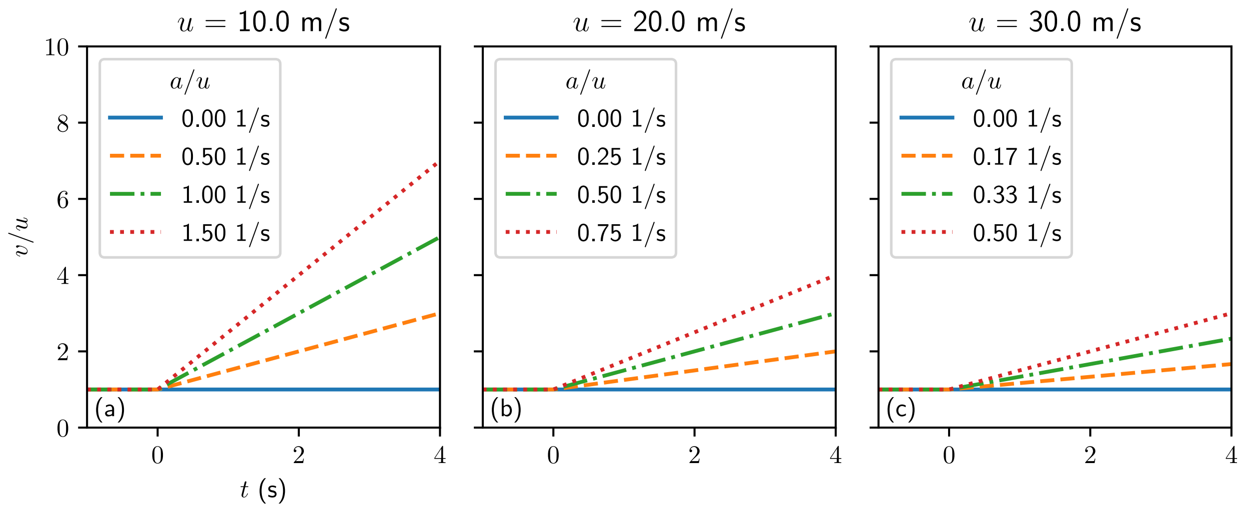

Figure 4.2: Same as Figure 4.1 but with v/u plotted instead of u. The legend, which shows the value of a/u, is now different for each panel.

plotted this way, since initially $v = u$, all the graphs have an initial condition equal to 1. And the slope of the lines correspond to the ratio a/u, which has dimensions of T^{-1}. Carefully observe the panels of Figure 4.2, and note that the graph corresponding to $a/u = 0.5$ 1/s is the same in all three panels. It is so because the ratio v/u satisfies

$$\frac{v}{u} = 1 + \frac{a}{u}t,$$

so that the slope of the lines is the value of a/u. This means that the curves in the three panels in Figure 4.2

are indeed identical for identical values of the parameter a/u, even if a and u may be different. Therefore, they can all be combined in a single graph, where different driving conditions are distinguished by different values of a/u. This is done in Figure 4.3(a).

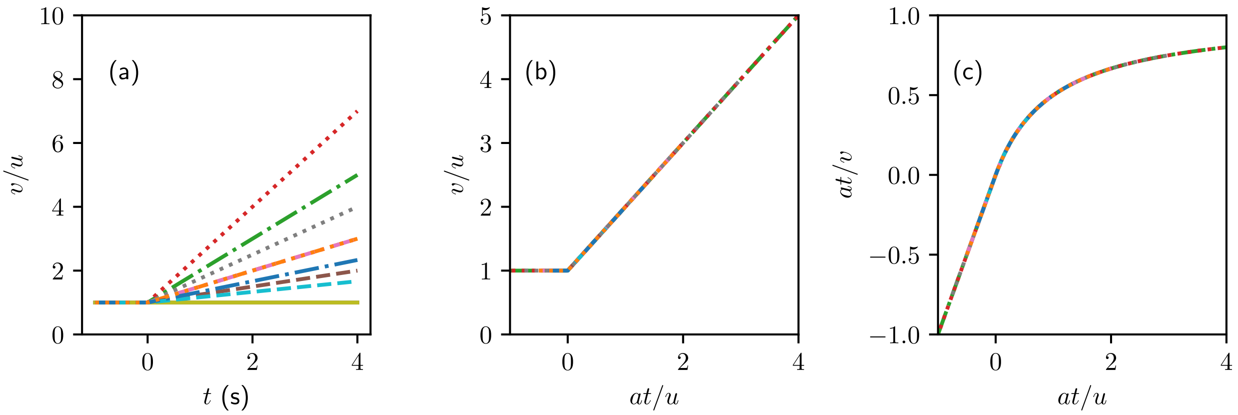

Figure 4.3: Single graphs that depict the velocity of an accelerating car. (a) The three panels of Figure 4.2 combined into one. The curves correspond to $a/u = 0,\ 0.17,\ 0.25,\ 0.33,\ 0.5,\ 0.75,\ 1.0,\ 1.5\ \ 1/\mathrm{s}$. (b) The data from the curves in (a) plotted as a function of at/u. (c) The same data as in (a) but with at/v plotted against at/u.

Note how rescaling the variables collapsed a family of curves on top of each other making them identical. This raises the question whether there is a way to further collapse the family of curves in Figure 4.3(a) into a single curve. After all, they have a similar fundamental shape. A moments thought will convince you that they can be reduced to a single curve if a different horizontal scale is used for each curve, i.e. a different time axis. One way of achieving this is by plotting v/u against at/u. Figure 4.3(b) shows this graph, which collapses all the curves into a single curve, resulting in a graph that contains all driving conditions. Both axes are now dimensionless and correspond to numbers identified earlier.

This example provides the first real indication of the power of Dimensional Analysis. It suggests that

Principle 4.2. The number of parameters in a problem is reduced by expressing the relationship in dimensionless form.

However, note that v/u versus at/u is not the only non-dimensionalization that will collapse all the curves. Figure 4.3(c) shows that plotting at/v versus at/u also collapses all the data on a single curve. This curve represents the relationship

$$\frac{at}{v} = \begin{cases} \frac{at}{u} & t < 0 \\ \dfrac{\frac{at}{u}}{\frac{at}{u}+1} & t > 0 \end{cases}. \tag{4.6}$$

There are in fact an infinite number of combinations that will collapse all the data. Any two independent combinations of the two dimensionless numbers v/u and at/u collapses the data. We will see that this non-uniqueness is a general property of non-dimensionalization of equations.

Purely from algebraic considerations, the collapse of data in Figure 4.3(b) contains the same data as in Figure 4.3(c). There is no algebraic reason to prefer one over the other. The two representations are algebraically equivalent. However, as humans, we prefer the representation in Figure 4.3(b). It is so because we implicitly treat the velocity of the accelerating car as primarily a function of time, and consider u and a as parameters. It was for this reason that we plotted time on the horizontal axis in Figure 4.1 and the different curves correspond to different u and a.

Therefore, it provides us greater insight when the quantity on the vertical axis is proportional to the dependent quantity of interest v, and the one on the horizontal axis is proportional to t. In this representation, it can be readily inferred that v is a constant for $t < 0$ and increases linearly with time for $t > 0$, which serves our intuitive understanding of the dynamics. Such insight is not available from the representation in Figure 4.3(c). And for such reasons of human insight, certain non-dimensionalizations are preferred over others. A similar observation is made in Question 4.3.

Principle 4.3. While the dimensionless parameters representing a problem are not unique, every representation is algebraically equivalent to every other. However, certain representations aid in providing insight into the problem.

Question 4.4. The height of a projectile launched vertically upwards with initial speed v from ground level $(y = 0)$ is given by

$$y = vt - \frac{1}{2}gt^2, \tag{4.7}$$

where t is the time since the launch. Figure 4.4 shows the height as a function of time for various v and g.

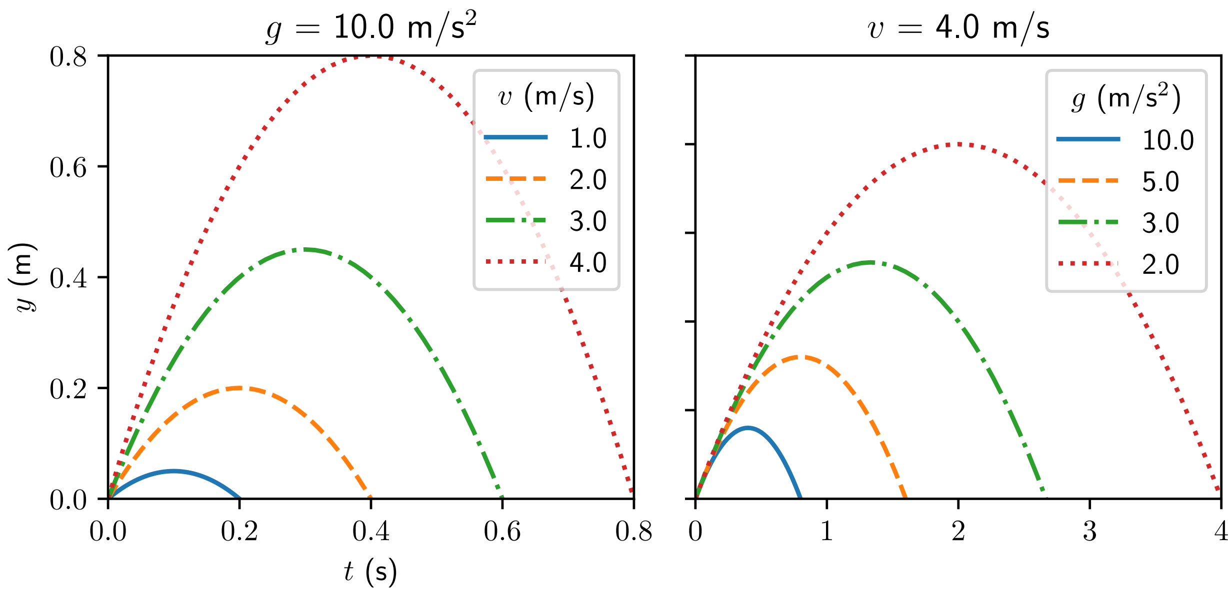

Figure 4.4: The height reached by a projectile launched vertically upwards as a function of time. (a) Varying v for a fixed g, and (b) Varying g for a fixed v.

Generate a dimensionless version of this graph and, if possible, collapse all these curves to a single master curve.

Answer 4.4. If equation 4.7 is non-dimensionalized by dividing by vt, we get

$$\frac{y}{vt} = 1 - \frac{gt}{2v}, \tag{4.8}$$

which suggests plotting $y/(vt)$ against gt/v. This is presented in Figure 4.5(a). All the curves in Figure 4.4 collapse to a single curve, a straight line. It is, however, unclear how to interpret this straight line. Instead consider the non-dimensionalization obtained from dividing equation (4.7) by v^2/g, which yields

$$\frac{gy}{v^2} = \frac{gt}{v} - \frac{1}{2}\left(\frac{gt}{v}\right)^2. \tag{4.9}$$

This leads to plotting $\tilde{y} = 2gy/v^2$ versus $\tilde{t} = gt/(2v)$ (the factors of two are included for better interpretation), which also collapses all the curves of Figure 4.4 into a single master curve. This curve is a parabola, which starts at $(0, 0)$ and returns to $y = 0$ at $gt/(2v) = 1$, after reaching a maximum height of $2gy/v^2 = 1$ halfway between. Thus, this curve can be interpreted as plotting the fraction of the maximum height reached by the projectile as a function of the fraction of its flight duration!

4.4 Dimensionless relationships

(The example of the falling column is from Dr John P Longley's Lecture Notes, where it was used to illustrate the principle of isolated dimension.)

Dimensional analysis is useful in reducing the complexity of the problem by reducing the number of independent variables whether or not the the relationship between those variables is known before-hand either in the form of charts or equations. The principle of the isolated dimension is instrumental in this process, so let us first learn it.

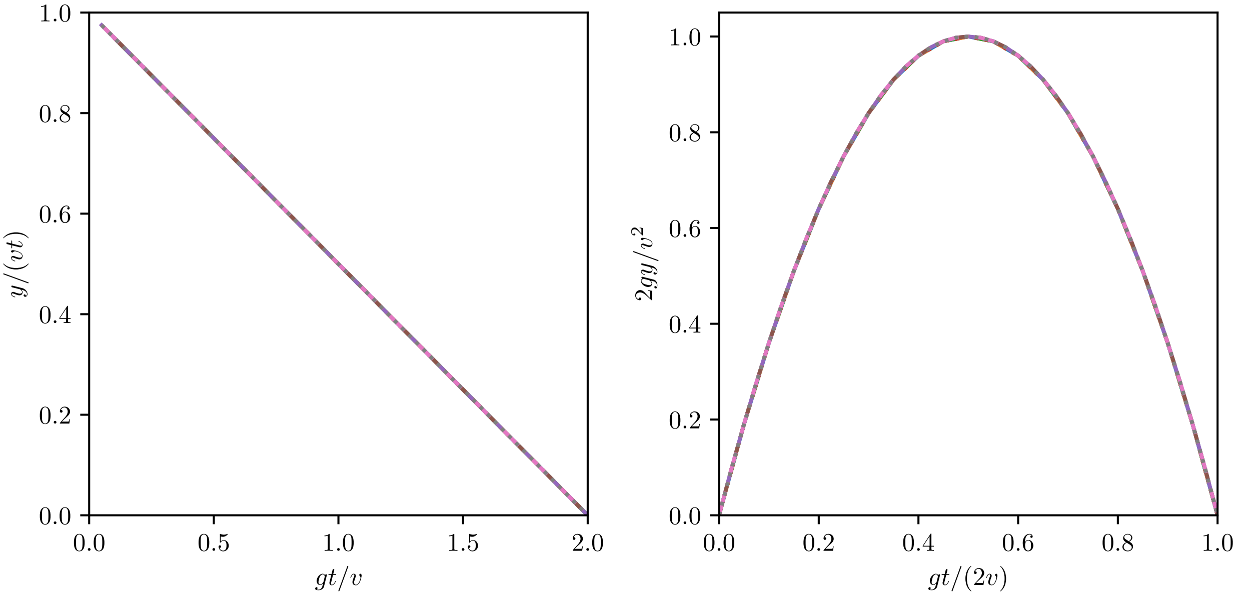

Figure 4.5: The trajectory of a vertically launched projectile plotted in terms of dimensionless variables. (a) Non-dimensionalization based on equation (4.7).

4.4.1 The principle of the isolated dimension

To illustrate this principle, consider the time taken for a column of height h with mass m to fall over from the vertical to the horizontal under the gravitational acceleration g. We write this relationship as

$$\begin{array}{rcl} \text{Parameter: } t & \text{depends on} & \{\ h, \quad g, \quad m \quad \} \\ \text{Dimensions: } T & & \{\ L, \quad LT^{-2}, \quad M \quad \} \end{array} \tag{4.10}$$

The dependent quantity t has dimensions of T, which are independent of M, and of the independent quantities, m includes dimensions of M, which none of the other independent parameters do. No combination of t, g and h can be combined with m to make a dimensionless group. Hence, because the functional relationship must be expressible in terms of dimensionless parameters alone, we must discard m from the relationship.

Thus, we really have

$$\begin{array}{rcl} \text{Parameter: } t & \text{depends on} & \{\ h, \quad g \quad \} \\ \text{Dimensions: } T & & \{\ L, \quad LT^{-2} \quad \} \end{array} \tag{4.11}$$

Principle 4.4. Isolated dimension: When an independent variable contains an isolated dimension, which cannot be combined with any other parameter to render it dimensionless, then the dependence on that variable must be dropped as a matter of dimensional consistency.

Advanced tip. In this example, we discarded the dependence on m as a redundant variable on the basis of dimensional consistency. While this conclusion is justified in this case, in general this may not be the only logical conclusion. In more complex problems, it may be argued that rather than remove a variable to ensure dimensional consistency, another variable (with appropriate dimensions) may be needed to obtain a complete statement of the physical law. In other words, the variable is perhaps not really isolated and the analysis contains an as of yet unidentified parameter(s) with overlapping dimensions.

This possibility leads to an important point that often arises in the application of dimensional analysis. How does one determine all the independent parameters that the dependent variable depends on? In more complex engineering situations, identifying the independent parameters is not obvious or easy. In fact, that is the most non-trivial step in the application of dimensional analysis, and a user's physical intuition is perhaps the strongest tool at their disposal for this purpose. The following question demonstrates this idea.

As an example, consider how one would identify the parameters on which the frequency of a stretched string depends. One might approach it purely empirically, and simply vary all the parameters one can think of. In this list should be the mass, diameter, length and elastic properties of the string, the tension in the string. Perhaps one might also suspect that gravitational acceleration and the thermal conductivity of the wire material are also relevant. In such an instance, the thermal conductivity contains the isolated dimension of temperature, and one would need to either eliminate it or identify another parameter with the dimensions of temperature to include in the analysis.

Let us pretend that we do not know the physics behind the vibrations of a stretched string.

Question 4.5. Identify the parameters on which the natural frequency of oscillation of a stretched string depends. Note that the discussion in this question is quite subjective and is intended to expose the reader to the complexity of the process.

Answer 4.5. One might approach the process purely empirically, and simply design experiments to vary all the parameters one can think of. In this list should be the mass, diameter, length and elastic properties of the string, the tension in the string. Perhaps one might also suspect that gravitational acceleration and the thermal conductivity of the wire material are also relevant. In such an instance, the thermal conductivity contains the isolated dimension of temperature, and one would need to either eliminate it or identify another parameter with the dimensions of temperature to include in the analysis.

There are a number of difficulties with this approach.

1. The experiments may be cumbersome to construct and conduct. It might be difficult to isolate only one parameter to vary at a time, while holding all others fixed so that it can be determined whether that parameter enters the dependence. For example, the mass of the string could be varied by changing the material of the string, but that also changes the elastic properties of the string.

2. Even if one succeeds in varying a single parameter at a time, experimental errors may creep in. Thus, it may not be possible to distinguish the possibility of a genuine dependence from that of experimental errors. This can perhaps be resolved in a statistical manner by conducting a large number of experiments.

Another approach is to develop a conceptual model that connects the independent parameters to the dependent one via a hypothesized physical process. In the case of the stretched string, one can invoke an analogy with a simple harmonic oscillator. The tension in the string is responsible for the spring of the oscillator and provides the restoring force, while the mass of the string represents the mass of the oscillator.

$$\begin{array}{rlll} \text{Parameter: } f & \text{depends on} & \{\, \tau, & m \,\} \\ \text{Dimensions: } T^{-1} & & \{\, MLT^{-2}, & M \,\} \end{array} \tag{4.12}$$

where, as before, τ is the tension in the string, m is its mass, and f it's natural frequency of oscillations. Recognizing that L is an isolated dimension in τ, the length l is included as an independent parameter as

$$\begin{array}{rllll} \text{Parameter: } f & \text{depends on} & \{\, \tau, & m, & l \,\} \\ \text{Dimensions: } T^{-1} & & \{\, MLT^{-2}, & M, & L \,\} \end{array} \tag{4.13}$$

In this manner, a combination of dimensional analysis and physical intuition is used to construct the parameter list.

Interested students who wishes to sharpen their skills in constructing parameter dependence lists may consider the following systems as challenging examples: (i) The power generated by a wind turbine, (ii) the rotation rate of a Crookes radiometer.

4.4.2 Recombination of parameters

We can now use the principle of isolated dimension to further simplify the dependence in Equation (4.10). Let us undertake the following rearrangement:

$$\begin{array}{rlll} \text{Parameter: } t & \text{depends on} & \{\, h, & g/h \,\} \\ \text{Dimensions: } T & & \{\, L, & T^{-2} \,\} \end{array} \tag{4.14}$$

Equation (4.14) represents the same dependence as in (4.11) in terms of h and g/h instead of h and g. Note that since we have retained independent dependence on h, the value of g can be determined from the ratio g/h. Knowledge of h and g/h implies the knowledge of h and g, so no information is lost by this rearrangement.

4.4.3 Iterating to eliminate dimensions

Amongst the many possible recombinations (e.g. g/h^2, h/g, h^2/g^3, etc.), note that Equation (4.14) has chosen a recombination which isolates the dimension L. Applying the principle of isolated dimension, we conclude that t cannot depend on h separately. That is,

$$\begin{array}{llll} \text{Parameter: } t & \text{depends on} & \{\, g/h & \} \\ \text{Dimensions: } T & & \{\, T^{-2} & \} \end{array} \tag{4.15}$$

We now continue with the recombination with the remaining variables by replacing the dependent parameter t with $t\sqrt{\frac{g}{h}}$. The dependent variable is not exempt from recombination. Applying the same logic as in §4.4.2, knowledge of $t\sqrt{\frac{g}{h}}$ and $\sqrt{\frac{g}{h}}$ implies knowledge of t.

$$\begin{array}{llll} \text{Parameter: } t\sqrt{\frac{g}{h}} & \text{depends on} & \{\, g/h & \} \\ \text{Dimensions: } 1 & & \{\, T^{-2} & \} \end{array} \tag{4.16}$$

The objective of this recombination is to eliminate the dimension of T from the dependent variable. This allows application of isolated dimension to conclude that the recombined dependent variable cannot depend on the remaining independent parameter. That is

$$\begin{array}{llll} \text{Parameter: } t\sqrt{\frac{g}{h}} & \text{depends on} & \{\, 1 & \} \\ \text{Dimensions: } 1 & & \{\, 1 & \} \end{array} \tag{4.17}$$

In this way, iterative application of recombination to isolate dimensions and then eliminate the isolated parameter reduces the number of independent parameters by rendering the dependence dimensionless. In this example, we conclude that $t\sqrt{\frac{g}{h}}$ is a constant C for all vertical columns

$$t\sqrt{\frac{g}{h}} = C. \tag{4.18}$$

4.5 More examples

To reinforce the concepts, let us consider a few more examples. The first we consider is the accelerating car, which appeared in Dr John P Longley's Lecture Notes.

Question 4.6. Non-dimensionalize the dependence

$$\begin{array}{llllll} \text{Parameter: } v & \text{depends on} & \{\, u, & a, & t & \} \\ \text{Dimensions: } LT^{-1} & & \{\, LT^{-1}, & LT^{-2}, & T & \} \end{array} \tag{4.19}$$

Answer 4.6. We begin by noticing that neither of the two dimensions L and T in the dependence are isolated, thus we must start by recombining the parameters to create isolated dimensions. Let us isolate the dimension of L by replacing v by v/u and a by a/u.

$$\begin{array}{llllll} \text{Parameter: } \frac{v}{u} & \text{depends on} & \{\, u, & \frac{a}{u}, & t & \} \\ \text{Dimensions: } 1 & & \{\, LT^{-1}, & T^{-1}, & T & \} \end{array} \tag{4.20}$$

Now that the dimension L is isolated, we can eliminate the dependence on u.

$$\begin{array}{lllll} \text{Parameter: } \frac{v}{u} & \text{depends on} & \{\, \frac{a}{u}, & t & \} \\ \text{Dimensions: } 1 & & \{\, T^{-1}, & T & \} \end{array} \tag{4.21}$$

Next we recombine with the intention of isolating the dimension T by replacing $\frac{a}{u}$ by $\frac{at}{u}$. Note here that we can also use $\frac{u}{at}$, and the result will be algebraically indistinguishable. However, to aid in intuition, we wish to construct our parameters such that they are proportional to t. Hence we prefer $\dfrac{at}{u}$ over $\dfrac{u}{at}$.

$$\begin{array}{lcl} \text{Parameter: } \dfrac{v}{u} & \text{depends on} & \{\ \dfrac{at}{u}, \quad t \quad \} \\ \text{Dimensions: } 1 & & \{\ 1, \quad T \quad \} \end{array} \tag{4.22}$$

Now that we have isolated the dimension T, we can eliminate the dependence on the variable t to yield

$$\begin{array}{lcl} \text{Parameter: } \dfrac{v}{u} & \text{depends on} & \{\ \dfrac{at}{u} \quad \} \\ \text{Dimensions: } 1 & & \{\ 1 \quad \} \end{array} \tag{4.23}$$

No further dimensions remain in this dependence to be eliminated. Hence, we conclude that $\dfrac{v}{u} = f\left(\dfrac{at}{u}\right)$ for some unspecified function f. What we have not obtained in the exact form of the relationship between v/u and at/u. However, the dimensionless relationship, v/u depends on only at/u, implies that it must be a single curve on a single graph. Thus we can determine everything about the accelerating car problem after a single experiment that produces the fundamental dimensionless graph.

Of course, we know from Question 4.1 that the functional form is $\dfrac{v}{u} = 1 + \left(\dfrac{at}{u}\right)$. The power of dimensional analysis is that we did not have to know this form *a priori* to simplify the dependence.

The role of dimensional analysis underneath the treatment of Questions 1.1-1.4 should now be transparent. Revisit those questions as a practice. We will treat Question 1.5, which appeared in Dr John P Longley's Lecture Notes, using methods from this chapter.

Question 4.7. Reduce the dependence from Question 1.5

$$\begin{array}{lcl} \text{Parameter: } v_3 & \text{depends on} & \{\ m_1, \quad m_2 \quad v_1 \quad v_2 \quad \} \\ \text{Dimensions: } LT^{-1} & & \{\ M, \quad M \quad LT^{-1} \quad LT^{-1} \quad \} \end{array} \tag{4.24}$$

to its dimensionless form.

Answer 4.7. Let us construct three recombined parameters m_2/m_1, v_2/v_1 and v_3/v_1.

$$\begin{array}{lcl} \text{Parameter: } \dfrac{v_3}{v_1} & \text{depends on} & \{\ m_1, \quad \dfrac{m_2}{m_1} \quad v_1 \quad \dfrac{v_2}{v_1} \quad \} \\ \text{Dimensions: } 1 & & \{\ M, \quad 1 \quad LT^{-1} \quad 1 \quad \} \end{array} \tag{4.25}$$

Both M and L/T are now isolated dimensions and they may be eliminated for dimensional consistency. However, note a peculiarity in this dependency. Clearly, L is an isolated dimension because it only occurs in v_1 and no other parameters in equation (4.25). But once v_1 is eliminated on account of L being isolated, then T is eliminated along with it. This double-elimination occurred because L and T did not appear independently in the parameters, but only in the fixed combination L/T. This leads to the dependence

$$\begin{array}{lcl} \text{Parameter: } \dfrac{v_3}{v_1} & \text{depends on} & \{\ \dfrac{m_2}{m_1} \quad \dfrac{v_2}{v_1} \quad \} \\ \text{Dimensions: } 1 & & \{\ 1 \quad 1 \quad \} \end{array} \tag{4.26}$$

We arrive at the same conclusion as of Question 1.5, but without knowing the underlying form of the relationship.

Let us look at the case of the vertical projectile from Question 4.4 to see the complexity involved in the various choices in forming dimensionless parameters that are algebraically equivalent.

Question 4.8. The instantaneous height of a projectile launched vertically upwards depends on the launch velocity, gravitational acceleration and time since launch as

$$\begin{array}{lcl} \text{Parameter: } y & \text{depends on} & \{\ v, \quad g \quad t \quad \} \\ \text{Dimensions: } L & & \{\ LT^{-1}, \quad LT^{-2} \quad T \quad \} \end{array} \tag{4.27}$$

Deduce the dimensionless relationship between these parameters, with the intuition that we seek y to be primarily a function of t.

Answer 4.8. We will explicitly refrain from (i) using t in any recombination with y, and (ii) retain t in the numerator of all recombinations involving it. Therefore, that leaves us with either v or g to combine with y to eliminate the dimension of L. Let us pick v (an we leave it as an exercise to the reader to pick g). Similarly, let us construct v/g.

$$\begin{array}{rlllll} \text{Parameter: } \dfrac{y}{v} & \text{depends on} & \{ \dfrac{v}{g}, & g & t & \} \\ \text{Dimensions: } T & & \{ T, & LT^{-2} & T & \} \end{array} \tag{4.28}$$

Now L is an isolated dimension in the parameter g, so that dependence can be dropped.

$$\begin{array}{rllll} \text{Parameter: } \dfrac{y}{v} & \text{depends on} & \{ \dfrac{v}{g}, & t & \} \\ \text{Dimensions: } T & & \{ T, & T & \} \end{array} \tag{4.29}$$

Since we cannot use t to remove the dimension of T from y, we will use v/g.

$$\begin{array}{rllll} \text{Parameter: } \dfrac{gy}{v^2} & \text{depends on} & \{ \dfrac{v}{g}, & \dfrac{gt}{v} & \} \\ \text{Dimensions: } 1 & & \{ T, & 1 & \} \end{array} \tag{4.30}$$

The dimension of T is now isolated in v/g, so that dependence can be dropped. We are then left with the dimensionless relationship

$$\begin{array}{rlll} \text{Parameter: } \dfrac{gy}{v^2} & \text{depends on} & \{ \dfrac{gt}{v} & \} \\ \text{Dimensions: } 1 & & \{ 1 & \} \end{array} \tag{4.31}$$

leading to $\dfrac{gy}{v^2} = f\left(\dfrac{gt}{v}\right)$, where $f(\cdot)$ is an unspecified function. We thus recover the conclusion of Question 4.4 without knowing the explicit form of the dependence.

4.6 Buckingham's Pi theorem

Buckingham's Pi theorem is named so because Buckingham denoted dimensionless groups of parameters by the Greek symbol for capital pi Π. The theorem sets an expectation for the number of dependent and independent dimensionless groups that make up a dimensionless form of a relationship based on the postulated dimensional form.

Principle 4.5. Let us assume we have N variables in circumstances where the statement

$$\text{Parameter: } p_1 \qquad \text{depends on} \quad \{ p_2, \quad p_3, \quad p_4, \quad \ldots \quad p_N \quad \}, \tag{4.32}$$

expresses a complete relationship between the p_1, p_2, $\ldots p_N$, which require M dimensions. According to principle of dimensional consistency, these can be expressed in dimensionless terms alone. Thus, equation (4.32) is equivalent to

$$\text{Parameter: } \Pi_1 \qquad \text{depends on} \quad \{ \Pi_2, \quad \Pi_3, \quad \Pi_4, \quad \ldots \quad \Pi_K \quad \}, \tag{4.33}$$

where there are $K \leq N$ dimensionless groups denoted by Π_1, Π_2, Π_3, $\ldots \Pi_K$. Then

$$K \geq N - M. \tag{4.34}$$

This result is not unexpected once we understand the method of elimination to non-dimensionalize dependencies. Here we present the proof, which appeared in Dr John Longley's Lecture Notes. In the production of the dimensionless groups, if a variable is discarded each time a dimension is eliminated, we find that the remaining number of dimensionless groups is

$$K = N - M$$

If, for any reason, eliminating a variable "costs" more than one dimension (which happened in Question 4.7 – when eliminating L, T is also eliminated), then

$$K > N - M$$

So that in general we get equation (4.34).

The following question demonstrate the theorem.

Question 4.9. Consider the collision between two point masses m_1 and m_2, initially moving at speeds v_1 and v_2, as in Question 1.5 and 4.7. Apply Buckingham's Pi theorem to the situation with M, L and T as the base dimensions.

Repeat the analysis with M, L and V (velocity) as the base dimensions.

Answer 4.9. There are five parameters in this dependence v_3, v_1, v_2, m_1 and m_2, so $N = 5$. If we take M, L and T as the base dimensions, then $M = 3$. So Buckingham's Pi theorem states

$$K \geq N - M = 5 - 3 = 2.$$

We expect at least two dimensionless variables.

Instead, in the MLV system, all the variables may be described using only the dimensions of M and V. The dimension of L is not needed. In this case, $N = 5$ as before, but M=2. Hence Buckingham's Pi theorem concludes that

$$K \geq N - M = 5 - 2 = 3.$$

Thus, we expect at least three dimensionless variables.

At first glance, the two conclusions may appear conflicting. If we did not know the result from Questions 1.5 and 4.7, it may not be clear which of the two ($K \geq 2$ or $K \geq 3$) applies in this case. The answer is that both are correct, and it is so because they are inequalities. But perhaps the more useful result is that $K \geq 3$, because it subsumes within it the possibility that $K \geq 2$, but not vice versa. And an additional observation that M and V are the minimum number of independent dimensions required to represent all parameters of the problem, implies that the inequality is, in fact, an equality, i.e. $K = 3$.

4.7 Alternative ways to form dimensionless relationships

The recombination-elimination is perhaps the most systematic method for identifying dimensionless form of parameters, there are other methods, perhaps more opaque but completely equivalent. We will illustrate these methods using the example of the falling column introduced in §4.4.1. The first one is the method of indices.

4.7.1 Indicial method

Reconsider the dependence

$$\begin{array}{lll} \text{Parameter: } t & \text{depends on} & \{\, h, \quad g, \quad m \,\} \\ \text{Dimensions: } T & & \{\, L, \quad LT^{-2}, \quad M \,\} \end{array} \tag{4.35}$$

We will then say that there exist exponents α, β and γ such that $th^\alpha g^\beta m^\gamma$ is dimensionless. The dimensions of $th^\alpha g^\beta m^\gamma$ are

$$T \times L^\alpha \times (LT^{-2})^\beta \times M^\gamma = M^\gamma L^{\alpha+\beta} T^{1-2\beta}. \tag{4.36}$$

Since we expect this combination to be dimensionless, i.e. with dimensions $M^0 L^0 T^0$, we obtain for the exponents three equations

$$\gamma = 0, \qquad \alpha + \beta = 0, \qquad \text{and} \qquad 1 - 2\beta = 0. \tag{4.37}$$

The unique values of the exponents that satisfy these equations simultaneously are $\alpha = -1/2$, $\beta = 1/2$ and $\gamma = 0$. Thus, we find that only one dimensionless combination is $t\sqrt{g/h}$ (and the m is eliminated).

This method is most useful when there is a single dimensionless parameter that can be constructed. When there is more than one, the solution to the equations for the exponents is not unique, so the user must make choices. We will not pursue this method further.

4.7.2 Inspection

Dimensionless parameters may also be constructed by inspection. After some experience, the user may spot that the combination of h/g has dimensions of T^2 and can use its square-root to form a dimensionless ratio in combination with another quantity with dimensions of T. One thus recovers $t\sqrt{g/h}$ as the dimensionless combination.

4.8 Conclusion

In this chapter, we examined how to construct dimensionless numbers and dimensionless relationships between parameters. The most reliable method for doing so is the method of recombination-elimination, but there also are other methods. The dimensionless combinations are not unique, and in fact any independent combination of dimensionless parameters is yet another set of dimensionless parameters that equally well describes the relationship. However, certain dimensionless combinations may be the best in aiding physical intuition of the users. Indeed, in this chapter it has become more and more clear that physical intuition is the most powerful tool at the user's disposal and a worthy goal to pursue with the help of dimensional analysis.

Chapter 5

Applications of Dimensional Analysis

It is nearly impossible to cover all possible applications of dimensional analysis because of the fundamental nature of its underlying principles. Here we will consider two applications.

5.1 Non-dimensional presentation of experimental data

The scope of the results of an experiment can be greatly expanded with the application of dimensional analysis. Consider the example of thrust generated by an aircraft propeller. Identical analysis also applies to marine propellers.

Propeller thrust

It is intuitive to deduce that thrust, f, generated by a propeller depends on its rotation rate, ω.

$$\text{Parameter: } f \quad \text{depends on} \quad \{\ \omega \quad \} \tag{5.1}$$

$$\text{Dimensions: } \frac{ML}{T^2} \quad \{\ T^{-1} \quad \} \tag{5.2}$$

However, this dependence must be incomplete because no dimensionless number canbe constructed from f and ω. Nor can the dependence on ω be eliminated by applying the principle of isolated dimension because it violates our physical intuition. Hence, the thrust must depend on additional parameters. For example, intuitively it is evident that the thrust must depend on the size of the propeller, say its diameter d.

The physics of propulsion with a propeller is as follows (consult a fluid dynamicist to clarify further, if needed). The surface of a propeller blade, as it spins, pushes the fluid back, and the force of reaction pushes the propeller forward. The amount of force needed to push the fluid back depends on its inertia, i.e. how heavy it is. This property is characterized by the fluid's density. For aviation applications, the density of air is approximately 1.2 kg/m^3. For fresh water, the density is 1000 kg/m^3, while for sea water in the North Sea it is 1030 kg/m^3, so they are quite similar.

The precise angle at which the propeller blade encounters the fluid depends on the speed of the boat and the linear speed of the various sections of the propeller. While the overall kinematics may be complex, it suffices to say that a change in either will influence the thrust. We know from general experience that a fan that spins faster generates a stronger flow. It is reasonable to consider that perhaps the same process operates in a propeller and the thrust depends on the rotation rate. It is also conceivable that the speed of the aircraft or the boat also influences the propeller thrust. This dependence is shown in Figure 5.1(a) using the data generated experimentally for one propeller by Johansson[1]. The data shows that, as the aircraft accelerates through the air, the thrust decreases. Based on this background, the thrust could be said to depend on d, ω, v and ρ.

The following table shows the variables, their SI units and their dimensions in the MLT system.

Quantity	Symbol	SI Unit	Dimension
Thrust	f	N = kg m/s^2	MLT^{-2}
Diameter	d	m	L
Angular velocity	ω	rad/s	T^{-1}
Ferry speed	v	m/s	LT^{-1}
Fluid density	ρ	kg/m^3	ML^{-3}

[1] Johansson, Marcus (2025). "Experimental Analysis of Rotating Propeller Performance in a Wind Tunnel." (Degree project), KTH, Stockholm, Sweden.

(The reader can try to work this example out in the FLT or the FLV or the FVT system, where F and V are the dimensions of force and velocity, respectively.) Note that radians are defined as the ratio of two lengths and, therefore, are dimensionless.

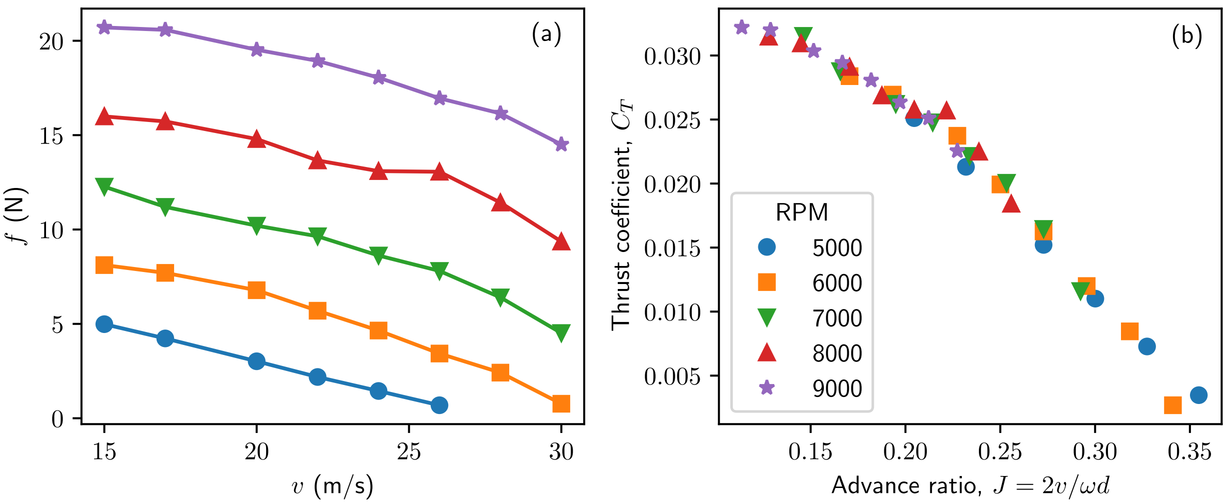

Figure 5.1: Thrust generated by the APC 11×10E propeller from the work by Marcus Johansson. (a) Thrust versus advance speed for difference rotation rates. (b) The dimensionless thrust coefficient C_T versus the advance ratio J.

Advanced tip. There is some flexibility in this choice. For designing a propeller, perhaps the thrust on the model could be specified but the rpm of the model propeller which generates the requisite thrust is to be determined using scale models. This changes whether the thrust is the dependent variable or the rotation rate. The reader is strongly encouraged to consider these possibilities independently.

Question 5.1. The thrust generated by a propeller depends on the following parameters.

$$\begin{array}{rlll} \text{Parameter: } f & \text{depends on} & \{\, d, \quad \omega, \quad v, \quad \rho \,\} \\ \text{Dimensions: } MLT^{-2} & & \{\, L, \quad T^{-1}, \quad LT^{-1}, \quad ML^{-3} \,\} \end{array} \tag{5.3}$$

Reduce this dependence to a dimensionless one.

Answer 5.1. We shall use the recombination-elimination method. Only f and ρ contain the dimension M, so let us introduce f/ρ instead of f.

$$\begin{array}{rlll} \text{Parameter: } \dfrac{f}{\rho} & \text{depends on} & \{\, d, \quad \omega, \quad v, \quad \rho \,\} \\ \text{Dimensions: } L^4T^{-2} & & \{\, L, \quad T^{-1}, \quad LT^{-1}, \quad ML^{-3} \,\} \end{array} \tag{5.4}$$

Equation (5.4) shows that the dimension M appears isolated in the parameter ρ and, therefore, the LHS may have no dependence on it. We can thus eliminate ρ as a parameter.

$$\begin{array}{rlll} \text{Parameter: } \dfrac{f}{\rho} & \text{depends on} & \{\, d, \quad \omega, \quad v \,\} \\ \text{Dimensions: } L^4T^{-2} & & \{\, L, \quad T^{-1}, \quad LT^{-1} \,\} \end{array} \tag{5.5}$$

Next we seek to eliminate the dimension L and we shall make use of the parameter d for this purpose. Let us construct $\dfrac{f}{\rho d^4}$ and $\dfrac{v}{d}$ instead of $\dfrac{f}{\rho}$ and v. Doing so leads to

$$\begin{array}{rlll} \text{Parameter: } \dfrac{f}{\rho d^4} & \text{depends on} & \{\, d, \quad \omega, \quad \dfrac{v}{d} \,\} \\ \text{Dimensions: } T^{-2} & & \{\, L, \quad T^{-1}, \quad T^{-1} \,\} \end{array} \tag{5.6}$$

In equation (5.6), the dimension L is isolated in the parameter d (by designing the previous recombination), and thus we can eliminate d from the dependence.

$$\begin{array}{rlll} \text{Parameter: } \dfrac{f}{\rho d^4} & \text{depends on} & \{\, \omega, \quad \dfrac{v}{d} \,\} \\ \text{Dimensions: } T^{-2} & & \{\, T^{-1}, \quad T^{-1} \,\} \end{array} \tag{5.7}$$

The last iteration is now evident. Recombine the parameters to construct $\frac{f}{\rho d^4 \omega^2}$ and $\frac{v}{\omega d}$ instead of $\frac{f}{\rho d^4}$ and $\frac{v}{d}$. This yields

$$\begin{array}{lcl} \text{Parameter: } \dfrac{f}{\rho d^4 \omega^2} & \text{depends on} & \{\ \omega, \quad \dfrac{v}{d\omega} \quad \} \\ \text{Dimensions: } 1 & & \{\ T^{-1}, \quad 1 \quad \} \end{array} \tag{5.8}$$

Doing so has isolated the dimension T in the parameter ω, which may now be eliminated, leading to the final dimensionless result.

$$\begin{array}{lcl} \text{Parameter: } \dfrac{f}{\rho d^4 \omega^2} & \text{depends on} & \{\ \dfrac{v}{d\omega} \quad \} \\ \text{Dimensions: } 1 & & \{\ 1 \quad \} \end{array} \tag{5.9}$$

The two dimensionless parameters thus identified are modified to match with convention in aerodynamics as follows. Define the thrust coefficient, C_T, as

$$C_T = \frac{f}{\frac{1}{2}\rho\omega^2 \left(\frac{d}{2}\right)^2 \left(\frac{\pi d^2}{4}\right)} = \frac{32}{\pi}\frac{f}{\rho\omega^2 d^4} \tag{5.10}$$

and the advance ratio, J, as

$$J = \frac{v}{(\omega d/2)} = \frac{2v}{\omega d}. \tag{5.11}$$

Note that C_T and J differ from the dimensionless groups in (5.9) only by factors of constants, so are consistent with dimensional analysis. The thrust coefficient signifies the dimensionless thrust and the advance ratio the dimensionless boat or aircraft speed.

The experimental data is plotted in terms of these dimensionless variables, which leads Figure 5.1(b). Notice how the data for different ω and v all collapse to a single curve.

This example illustrates how the reduction in number of parameters provided by dimensional analysis is utilized in presenting experimental data. One can readily see the redundancy in the data, so engineers and scientists usually exploit dimensional analysis to reduce the number of experiments.

The following example on the dip in a hanging cable from Dr John P Longley's Lecture Notes further illustrates the point. The data and the solution are also from Dr Longley's notes.

Dip in a hanging cable

An inextensible, but otherwise flexible cable is suspended between two points at the same vertical elevation but a horizontal distance l apart. The ability to predict the dip clearly has many applications, for example in constructing overhead power distribution lines on electrified railway tracks. The dip can be reduced by pulling on the cable with a force f to stretch them taut as shown in Figure 5.2. How much does it dip at the midpoint for a given force f?

Dr Longley designed an experiment with a cable that can be suspended between two supports mounted on a desktop, essentially reproducing the setup in Figure 5.2. By varying the tension force f and measure the dip d for a fixed span l, the results he obtained are shown in Table 5.1 and plotted in Figure 5.3 below.

Tension force	**Cable dip**
f (N)	d (mm)
0.25	59
0.49	29
0.74	20
0.98	15
1.23	12

Table 5.1: Data from the experiment varying f and measuring d for a fixed $l = 2.4$ m and $w = 0.02$ N/m.

The data can be used to predict the dip for a given tension force, perhaps with a little interpolation if the force did not happen to coincide exactly with the values in the dataset. One may think that the results of this

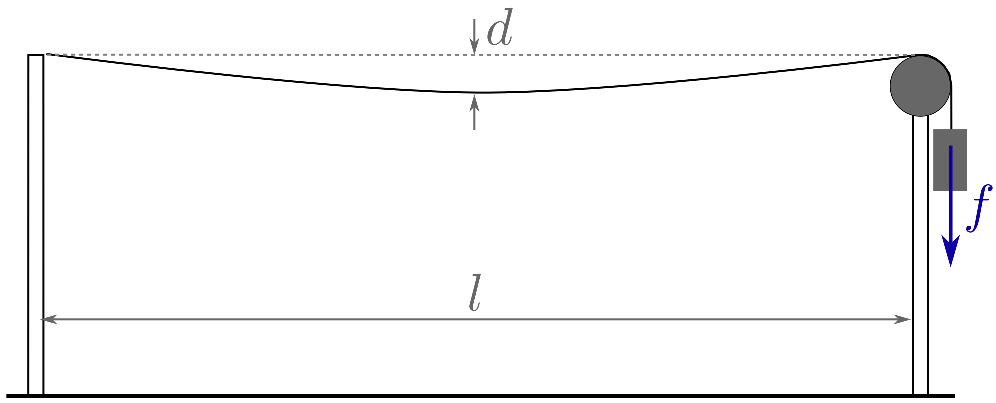

Figure 5.2: A schematic depiction of the setup showing a cable suspended between two fixed points at equal elevation and distance l apart, with an force f stretching it. The weight of the cable per unit length is w. The objective is to characterize the dip in its elevation in the middle.

experiments are only useful to one case – the case of the cable used by Dr Longley, with the length spanning the desk. But Dr Longley has been teaching Dimensional Analysis since 1993! He readily anticipated that he could use this data to predict the dip in any inextensible but otherwise flexible cable *using Dimensional Analysis.*

Let us first determine the dependence in dimensionless form. To enable us to determine the parameters, we start with the hypothesized dependence

$$\begin{array}{rcl} \text{Parameter: } d & \text{depends on} & \{\, l, \quad f \quad \} \\ \text{Dimensions: } L & & \{\, L, \quad F \quad \} \end{array} \tag{5.12}$$

where we have chosen the FLT system of base dimensions. Here, we immediately note that f contains the isolated dimension F. Therefore, either f must be eliminated or another parameter containing the dimension of F is missing and must be introduced. The experimental data shows the dependence on f, so it cannot be eliminated, so we must be missing a parameter. Intuitively, we realize that for the same applied force, a heavy cable would dip more than a lighter cable. Afterall, the applied force acts to somehow counteract the force of gravity, so the weight of the cable must enter the dependence. Dr Longley introduces the equivalent parameter, the weight per unit length of the cable, w, thus modifying the hypothesized dependency as

$$\begin{array}{rcl} \text{Parameter: } d & \text{depends on} & \{\, l, \quad w, \quad f \quad \} \\ \text{Dimensions: } L & & \{\, L, \quad FL^{-1}, \quad F \quad \} \end{array} \tag{5.13}$$

Afterall, if w is known then the cable weight is wl.

Let us quickly apply Buckingham's Pi theorem 4.5 to determine the expected number of dimensionless parameters. Here $N = 4$ and $M = 2$ (because only F and L are independent dimensions representing the parameters), so we expect $K \geq 4 - 2 = 2$ independent dimensionless parameters.

We can first recombine f and w into f/w and f, so the modified parameters are

$$\begin{array}{rcl} \text{Parameter: } d & \text{depends on} & \{\, l, \quad f/w, \quad f \quad \} \\ \text{Dimensions: } L & & \{\, L, \quad L, \quad F \quad \} \end{array} \tag{5.14}$$

and isolate the dimension F. (Note that here we made the tacit decision to retain f in the numerator so that the resulting dimensionless number can be interpreted as a dimensionless version of the stretching force.) The parameter with the isolated dimension can now be eliminated.

$$\begin{array}{rcl} \text{Parameter: } d & \text{depends on} & \{\, l, \quad f/w \quad \} \\ \text{Dimensions: } L & & \{\, L, \quad L \quad \} \end{array} \tag{5.15}$$

We can then further recombine the remaining parameters into d/l, l and $f/(wl)$ to get

$$\begin{array}{rcl} \text{Parameter: } \dfrac{d}{l} & \text{depends on} & \{\, l, \quad \dfrac{f}{wl} \quad \} \\ \text{Dimensions: } 1 & & \{\, L, \quad 1 \quad \} \end{array} \tag{5.16}$$

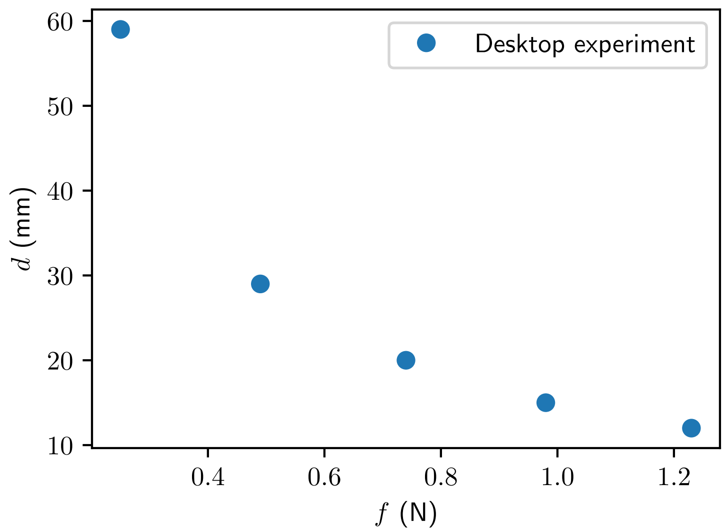

Figure 5.3: Data from Table 5.1 plotted as f versus d.

Doing so isolates the dimension L in the parameter l, which can now be eliminated. Finally, we are left with

$$\begin{array}{rlll} \text{Parameter:} & \dfrac{d}{l} & \text{depends on} & \left\{ \dfrac{f}{wl} \right\} \\ \text{Dimensions:} & 1 & & \{ 1 \} \end{array} \tag{5.17}$$

and the functional dependence is

$$\frac{d}{l} = F\left(\frac{f}{wl}\right), \tag{5.18}$$

for some unspecified function $F(\cdot)$. Equation (5.18) confirms the result from Buckingham's Pi theorem that there are two remaining dimensionless parameters.

Let us now interpret the dimensionless parameters. The first one d/l is simply the dip expressed as a fraction of the span. The second one $f/(wl)$ is nothing more than the stretching force written as a multiple of the cable weight. Intuitively, the amount of force needed to lift the cable must be related to how heavy it is, and that is what this dimensionless relation reflects.

We can now replot the data from the experiment (Table 5.1) in dimensionless terms. It is done in Fig. 5.4 and let us label this dataset as the Desktop experiment.

To demonstrate the value of presenting experimental data in dimensionless form, Dr Longley conducts another experiment using two different cables and across a different span.

	Weight per length w (N/m)	Dip in the cable d (mm)	$\frac{d}{l}$	$\frac{f}{wl}$
Blue cable	0.52	210	0.024	5.2
White cable	0.10	40	0.004	27.2

Table 5.2: Data from two further experiments with two different cables.The new span for these two experiments is $l = 8.9$ m. The tension force is $f = 24.2$ N.

If the premise of this analysis holds, then the new experimental data should agree with the data plotted in Figure 5.4. The astute reader can determine the success of Dr Longley's application of Dimensional Analysis by glancing at Figure 5.5.

Can now do Examples Paper Q7.

5.2 Scale models

Building and testing scale models is a common approach in engineering to learn essential underlying principles and aid in design. The fundamental basis of scale models lies in dimensional analysis. It is dimensional analysis

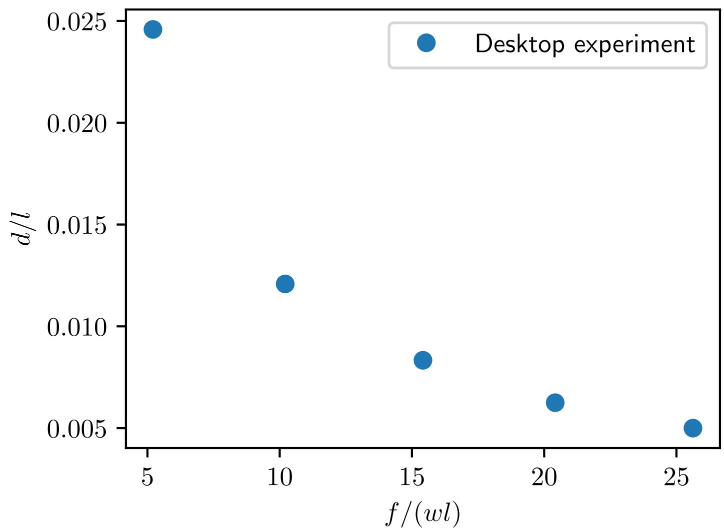

Figure 5.4: Dimensionless version of the data from the experiment presented in Table 5.1.

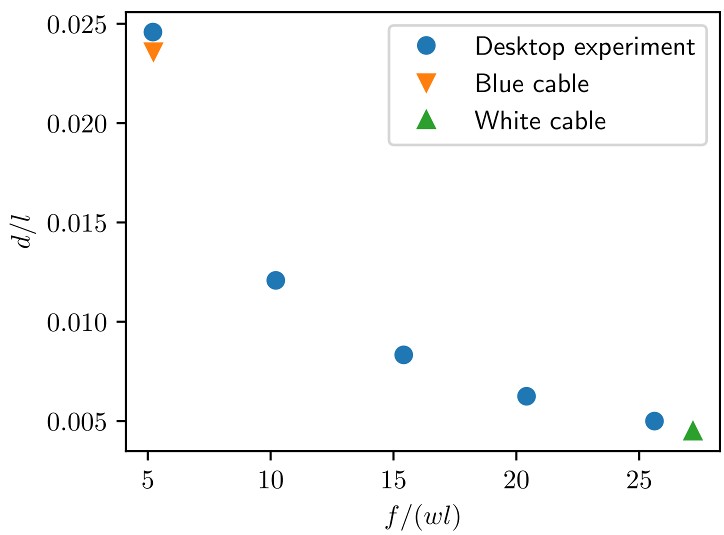

Figure 5.5: Comparison of the results of multiple experiments with the hanging cable. This graph includes data from Tables 5.1 and 5.2.

that enables testing of scaled models of Formula One automobiles and aircraft in wind tunnels. We already saw in §5.1 how experiments at different scales are related to each other through dimensional analysis. Here we take that observation one step further and design scale models of full-scale systems to test their performance and aid in design. For the purpose of scale modeling, understanding of two concepts is crucial: (i) geometric similarity, and (ii) dynamic similarity.

The following example illustrates the principles.

Question 5.2. (from Dr John P Longley's Lecture Notes)
A beam of length l with a rectangular cross section (breadth b, depth d), is simply supported on its two ends. A vertical force f is applied to its midpoint to cause it to dip vertically downwards by δ. Determine the dimensionless dependence of δ on the independent variables. (The elastic modulus of the material of the beam is E and has dimensions identical to pressure.)

Answer 5.2. We start with the dimensional dependence as

$$\begin{array}{llllllll} \text{Parameter: } \delta & \text{depends on} & \{\, f, & E, & l, & d, & b & \} \\ \text{Dimensions: } L & & \{\, F, & FL^{-2}, & L, & L, & L & \} \end{array} \tag{5.19}$$

Here we work in the FLT base system of dimensions because it reveals most clearly that there are only two independent dimensions (F and L). We can deduce by inspection that f/El^2 is dimensionless, and so are the ratios δ/l, d/l and b/l.

$$\begin{array}{rlcl} \text{Parameter:} & \dfrac{\delta}{l} & \text{depends on} & \left\{ \dfrac{f}{El^2}, \quad \dfrac{d}{l}, \quad \dfrac{b}{l} \quad \right\} \\ \text{Dimensions:} & 1 & & \{\, 1, \qquad 1, \quad 1 \quad \} \end{array} \tag{5.20}$$

The diligent student is asked to verify this result before proceeding.

5.2.1 Geometric similarity

A scale model is a replica of the the original in all respects except the overall size. Every dimension of the model is "scaled down" (or up, as the case may be) by a fixed factor. The concept of geometric similarity captures the shape of the object, while letting the overall size vary.

> **Principle 5.1.** (from Dr John P Longley's Lecture Notes)
> Two objects are geometrically similar when one object is geometrically an exact scale model of the other.

In Question 5.2, if a beam is a scale model of the original, then all their lengths would be reduced by the same factor, thereby leaving the dimensionless numbers b/l and d/l identical to the original. Therefore, for a given set of geometrically similar beams, the two geometric ratios are fixed, i.e. they do not vary. Then, the only variation in equation (5.20) arises from the applied force, leading to

$$\begin{array}{rlcl} \text{Parameter:} & \dfrac{\delta}{l} & \text{depends on} & \left\{ \dfrac{f}{El^2} \quad \right\} \\ \text{Dimensions:} & 1 & & \{\, 1 \qquad \} \end{array} \tag{5.21}$$

5.2.2 Dynamic similarity

The matching of geometry is necessary but not sufficient for a successful exercise of scale modeling. It is also necessary to have dynamic similarity for this purpose.

> **Principle 5.2.** (from Dr John P Longley's Lecture Notes)
> Two situations are dynamically similar when all independent dimensionless parameters have identical values. As a consequence, the value of the dependent dimensionless variable will also be identical.

The following question extends Question 5.2 to illustrate this principle.

Question 5.3. (from Dr John P Longley's Lecture Notes)
A one-tenth scale model of a beam is constructed from the same material as the original for the purposes of testing. If the original beam is expected to esperience a force of 10000 N, what force should be applied to the scale model to maintain dynamic similarity?

Answer 5.3. The question statement says the scale model is one-tenth scale, i.e. each dimension is 1/10 of full-size model. This ensures geometric similarity. The only remaining independent dimensionless parameter is $\dfrac{f}{El^2}$. We must ensure that this parameter has the same value in the scale model that it has for the full-size model.

$$\frac{f_{\text{fs}}}{El_{\text{fs}}^2} = \frac{f_{\text{sm}}}{El_{\text{sm}}^2}. \tag{5.22}$$

Here the subscript fs stands for full-scale and sm stands for scale model. Since we know $l_{\text{sm}} = l_{\text{fs}}/10$, equation (5.22) implies $f_{\text{sm}} = f_{\text{fs}}/100 = 100$ N.

5.2.3 A ship's propeller

Dr John P Longley constructs the following example for the design of a ships propeller to illustrate the use of scale models in engineering design. The same principle also applies to an airplane propeller.

A manufacturer of a ship's propellers has been commissioned to produce a propeller for a ferry which will operate in the North Sea. It has been determined independently that to cruise at a speed of 12 m/s, the ferry will need a thrust of 25 kN. The manufacturer has just produced a new but untried shape of propeller. For the ferry, the new propeller should have a diameter of 0.6 m and should rotate at 800 rpm.

In order to check the propeller design a $1/10^{\text{th}}$ scale model is to be built and tested. The model propeller (with diameter 0.06 m) will rotate at 8000 rpm and will be tested in fresh water. The model is to be tested under conditions which simulate the ferry cruising condition.

We need to know:

1. the speed of the ferry, and

2. the thrust that the model should produce.

Dr Longley suggests that we use (5.9) and exploit dynamic similarity.

The dimensionless analysis has identified two dimensionless parameters and it is a sound practice to interpret them for developing intuition. The dependent parameter is $\dfrac{f}{\rho d^4\omega^2}$, which consists of f, ρ, d and ω. Dynamic similarity between propellers of two scales implies that

$$\left(\frac{f}{\rho\omega^2 d^4}\right)_{\text{sm}} = \left(\frac{f}{\rho\omega^2 d^4}\right)_{\text{fs}} \qquad \Longrightarrow \qquad f_{\text{sm}} = f_{\text{fs}} \times \frac{\rho_{\text{sm}}}{\rho_{\text{fs}}} \times \left(\frac{d_{\text{sm}}}{d_{\text{fs}}}\right)^4 \times \left(\frac{\omega_{\text{sm}}}{\omega_{\text{fs}}}\right)^2 \tag{5.23}$$

It follows from (5.23) and stands to reason that a propeller with a larger diameter (greater d) generates more thrust, a propeller rotating faster (greater ω) generates more thrust, and a propeller pushing a heavier fluid (greater ρ) generates more thrust.

Similarly, the independent parameter $v/(\omega d)$ is interpreted as the speed of the propeller tip (ωd) relative to the boat speed (v). It represents the distance the boat (and with it the propeller) moves forward within one rotation of the propeller, and evokes the image of a screw. The mechanism of the propeller can now be interpreted as that of a screw pushing through a fluid. It is for this reason that a propeller is also colloquially referred to as a screw in nautical and aeronautical circles.

We now insist on *dynamically similarity* between the two scales.

The independent parameters

There is only one independent parameter, viz. $\dfrac{v}{\omega d}$. Matching the parameter on the two scales implies

$$\left(\frac{v}{\omega d}\right)_{\text{fs}} = \left(\frac{v}{\omega d}\right)_{\text{sm}} \qquad \Longrightarrow \qquad v_{\text{sm}} = v_{\text{fs}} \times \frac{\omega_{\text{sm}}}{\omega_{\text{fs}}} \times \frac{d_{\text{sm}}}{d_{\text{fs}}} = 12 \text{ m/s} \times \frac{8000}{800} \times \frac{0.06 \text{ m}}{0.6 \text{ m}} = 12 \text{ m/s}. \tag{5.24}$$

The model ferry should travel at 12 m/s.

The dependent parameter

Matching the dependent parameter between the two scales implies from (5.23)

$$f_{\text{sm}} = 25000 \text{ N} \times \frac{1000}{1030} \times \left(\frac{0.06}{0.6}\right)^4 \times \left(\frac{8000}{800}\right)^2 = 242.7 \text{ N}. \tag{5.25}$$

5.3 Conclusion

Geometric and dynamic similarity between different scales of the same system allows us to relate their behavior to each other using dimensional analysis. We saw in this chapter how to exploit this principle. Let us conclude with the following maxim by Dr John P Longley:

Principle 5.3. It is a general principle that the deeper the physical understanding at the start of the problem, the more likely that dimensional analysis will yield a significant result.